

CH # 2

1

Numerical Sequences

Series

Intuitive (of ideas) obtained by feelings rather than by considering facts:

Eventually # adv at the end of a period of time or a series of events or beyond some point of time.

Primitive # belonging to an early stages in the development.

ϵ -Neighbourhood

In any metric space (X, d) and $x_0 \in X$, open ball $B(x_0; \epsilon)$ is called an ϵ -neighbourhood (or sometimes neighbourhood).

If $x_0 \in \mathbb{R}$, then the open interval $(x_0 - \epsilon, x_0 + \epsilon)$ is called an ϵ -neighbourhood of x_0 . We denote neighbourhood by $N(x_0; \epsilon)$ or $N_\epsilon(x_0)$.

Deleted Neighbourhood

The set $(x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon)$ is called a deleted neighbourhood of x_0 because x_0

deleted from $N_{\in}(x_0)$. It is denoted by $N_{\in}(x_0)$

Sequence

A sequence in a set S is a function whose domain is the set N of natural numbers and range is a subset of the set S .

Real Sequence

A sequence of real numbers (or a sequence in R) is a function whose domain is the set N of natural numbers and range is a subset of the set R of real numbers.

Notations

(1) # If $x: N \rightarrow R$ is a sequence, we usually denote the value of x at n by symbol x_n instead of the function notation $x(n)$. $x_1, x_2, \dots, x_n, \dots$ are called terms of sequence.

(2). The sequence $x: N \rightarrow R$ is denoted by $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$ or $\langle x_n \rangle$ or $(x_n: n \in N)$ or (x_n) .

(3). Notation $\langle x_n \rangle$ or $(x_n: n \in N)$ or $\{(n, f(n)): n \in N\}$ is used to that terms

the sequence produced by the
 # The set of distinct terms
 called the range and is denoted
 $\{x_n : n \in \mathbb{N}\}$ and values in the range
 not ordered.

Since in a sequence $\{x_n\}, n \in \mathbb{N}$
 is an infinite set. Therefore the range
 terms of a sequence is always infinite.
 range of sequence may be a finite set.
 if $x_n = (-1)^n$ then $\{x_n\} = \{-1, 1\}$

The range of sequence $= \{-1, 1\}$ is finite.
 # Sequences are often defined by the
 a formula for the n th term x_n .
 # Sometimes it is convenient to write
 terms of a sequence in order starting
 the rule of definition is $x_m = x_n$

The m th and n th terms
 for $m \neq n$ are treated distinct
 i.e. the terms occurring at different
 positions are treated as distinct terms
 they have the same value.

$\dots\}$
 finite.
 ing
 when
 in
 $x_m = x_n$
 positions
 enough

Constant Sequence

A sequence $\{x_n\}$ defined by $x_n = c \in \mathbb{R}$
 $\forall n \in \mathbb{N}$ is called a constant sequence.

OR
A sequence whose range is a singleton
is called a constant sequence.

Mathematical Formula Notation

For Sequence

Many sequences may be defined by some
mathematical rule by Two ways.

(a) By an explicit formula. (b) by a recursion
or inductive or iterative formula.

(a) Explicit Formula

A sequence may be defined by giving an
explicit formula for the n th term. e.g

(1) $a_n = \frac{1}{n}$

(2) $a_n = \frac{n}{n+1}$

(3) $a_n = (-1)^{\frac{n}{n+1}}$

$a_n = 3$

(b) Recursive Formula

Sometimes sequences are defined by specifying
(clearly giving) one or more initial terms and
by giving a formula that relates each subsequent
(next coming) term to the previous terms. Such

Sequences are said to be defined recursively or inductively or iteratively and the defining formula is called a recursion formula or inductive formula.

A sequence defined OR by a formula for the n th term in terms of one or several previous terms with some initial terms specified clearly.

Examples #

1) $a_{n+2} = a_{n+1} + a_n$ $a_1 = 1$ $a_2 = 1$
 $1, 1, 2, 3, 5, 8, \dots$

2) # $a_n = \frac{a_{n+1}}{2}$ $n \geq 1$ $a_0 = 1$

3) # $a_n = n \cdot a_{n-1}$ $a_1 = 1$

4) # Fibonacci Sequence.

$a_{n+1} = a_n + a_{n-1}$ $a_1 = 1, a_2 = 1$

$1, 1, 2, 3, 5, 8, \dots$

Thus each term after the 1st two terms is the sum of its two immediate previous terms.

5) The sequence $\{2n\}$ of even numbers can be defined by $x_1 = 2$ $x_{n+1} = x_n + 2$.

OR by $y_1 = 2$ $y_{n+1} = y_1 + y_n$

6

Note not every sequence has a simple formula or even any formula at all. e.g. $\{d_n\}$, where d_n is n th digit in lto decimal representation of π

Bounded Above Sequence

A sequence $\{x_n\}$ is said to be bounded above if \exists a real no K such that

$$x_n \leq K \quad \forall n \in \mathbb{N}$$

i.e if the range of the sequence is bounded above.

Bounded Below Sequence

A sequence $\{x_n\}$ is said to be bounded below if \exists a real no k such that

$$x_n \geq k \quad \forall n \in \mathbb{N}$$

i.e if the range of the sequence is bounded below.

Bounded Sequence

A sequence is said to be bounded if it is bounded above as well as below:

Thus a sequence $\{x_n\}$ is bounded if \exists two numbers k & K ($k \leq K$) such that

23

\mathbb{Z}

$$k \leq x_n \leq K \quad \forall n \in \mathbb{N}$$

i.e if the range set $\{x_n : n \in \mathbb{N}\}$ is bounded.

A sequence that is not bounded is said to be unbounded sequence.

Unbounded Above

A sequence $\{x_n\}$ is said to be unbounded above if it is not bounded above i.e if for every real no $K \exists m \in \mathbb{N}$ s.t.

$$a_m > K$$

Unbounded Below

A sequence $\{x_n\}$ is said to be unbounded below if it is not bounded below i.e. if for every real no $k \exists m \in \mathbb{N}$ s.t.

$$a_m < k$$

Examples

(1) # The sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$ is bounded. because $0 < a_n \leq 1$.

(2) # The sequence $\{a_n\}$ defined by $a_n = n$ is bounded below by 1 because $a_n \geq 1 \quad \forall n \in \mathbb{N}$. It is not bounded above because \exists no real no K such that

$$a_n \leq K \quad \forall n \in \mathbb{N}$$

(3) # The sequence $\{(-1)^n\}$ is bounded because.

$$-1 \leq a_n \leq 1 \quad \forall n \in \mathbb{N} \quad \text{above}$$

(4) # The sequence $\{-n\}$ is bounded ~~below~~ because $a_n \leq -1 \quad \forall n \in \mathbb{N}$ & is not bounded below

(5) # Every constant sequence is bounded.

(6) # The sequence $\{a_n\}$ defined by $a_n = (-1)^n \cdot n$ is neither bounded above nor bounded below.

Theorem # A sequence $\{a_n\}$ is bounded.

iff \exists a +ve real no M such that

$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

Proof # Necessary Condition

Let $\{a_n\}$ be bounded. Then \exists two real numbers h & k such that

$$h \leq a_n \leq k \quad \forall n \in \mathbb{N}$$

Let $M = \max\{|h|, |k|\}$. Then.

$$|h| \leq M \quad \& \quad |k| \leq M.$$

$$\Rightarrow -M \leq h \leq M \quad \& \quad -M \leq k \leq M.$$

$$\Rightarrow -M \leq h \leq a_n \leq k \leq M \quad \forall n \in \mathbb{N}$$

$$\Rightarrow -M \leq a_n \leq M \quad \forall n \in \mathbb{N}$$

$$\Rightarrow |a_n| \leq M \quad \forall n \in \mathbb{N}$$

Sufficient Condition #9

Let M be the real no such that $\forall n \in \mathbb{N}$
 $|a_n| \leq M$

Then $-M \leq a_n \leq M \quad \forall n \in \mathbb{N}$

$\Rightarrow \{a_n\}$ is bounded.

Note The above Theorem is used as definition of bounded sequence.

Theorem # If $\{a_n\}$ & $\{b_n\}$ are bounded sequences and c is a real no, then

(a) $\{a_n + b_n\}$ is bounded.

(b) $\{a_n b_n\}$ is bounded.

(c) $\{c a_n\}$ is bounded.

Proof # $\because \{a_n\}$ & $\{b_n\}$ are bounded.
 $\therefore \exists$ +ve numbers M_1 & M_2

such that $|a_n| \leq M_1$ & $|b_n| \leq M_2 \quad \forall n \in \mathbb{N}$.

$\Rightarrow |a_n| + |b_n| \leq M_1 + M_2 = M \quad \forall n$

Now $|a_n + b_n| \leq |a_n| + |b_n| \leq M \quad \forall n \in \mathbb{N}$.

$\Rightarrow \{a_n + b_n\}$ is bounded.

(b) $|a_n b_n| = |a_n| |b_n| \leq M_1 M_2 = M_3 \quad \forall n$

$\Rightarrow \{a_n b_n\}$ is bounded.

(c) # $|c a_n| = \frac{8}{10} |a_n| = k |a_n| = M_1 = M_2 \quad \forall n$
 $\Rightarrow \{c a_n\}$ is bounded.

Eventual Property of a Sequence.

If the terms of a sequence do not have a certain property from start but have that property from some point (from some term) on, then the sequence has that property eventually.

If a sequence $\{a_n\}$ fulfils some property P eventually, then mathematically we say that \exists an integer $n_1 \in \mathbb{N}$ such that

if $n \geq n_1$, $\{a_n\}$ satisfies property P .

Limit of a Sequence & Convergence

A sequence $\{a_n\}$ in \mathbb{R} is said to converge to $l \in \mathbb{R}$ or l is a limit of $\{a_n\}$ if for every $\epsilon > 0$ \exists a natural no $n_1(\epsilon)$ such that

$$|a_n - l| < \epsilon \quad \forall n \geq n_1$$

If so we write $\lim_{n \rightarrow \infty} a_n = l$ or $\lim a_n = l$.
 or $a_n \rightarrow l$

If a sequence has a limit, then sequence is convergent, if it has no limit, the sequence is dgt.

Note (1) A real no l is a limit of sequence $\{a_n\}$ if given $\epsilon > 0$, all but a finite no of terms of $\{a_n\}$ lie within ϵ of l .

$$(2) \quad |a_n - l| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n \in (l - \epsilon, l + \epsilon) \quad \forall n \geq n_1$$

\Rightarrow given any $\epsilon > 0$, all the terms of the sequence, except the 1st $n_1 - 1$ terms lie in the interval $(l - \epsilon, l + \epsilon)$. The 1st $n_1 - 1$ terms may be scattered anywhere. The number $n_1 - 1$ of terms left out of the interval $(l - \epsilon, l + \epsilon)$ depends upon the size of ϵ . The smaller the size of ϵ , the larger will be the no of terms left out of $(l - \epsilon, l + \epsilon)$.

(2) We say that a sequence $\{a_n\}$ ~~has~~ ultimately has a certain property if \exists a no n_1 such that sequence $\{a_n\}$ satisfies that property for $n \geq n_1$. A sequence $\{a_n\}$ converges to l if the terms of $\{a_n\}$ are ultimately in every ϵ -neighbourhood of l .

(3) With the Language of Nbhd.
A sequence $\{a_n\}$ converges to number l if for each ϵ -nbhd $N_\epsilon(l)$ of l all but a finite no of terms of $\{a_n\}$ belong to $N_\epsilon(l)$

Examples

(a) $\lim_{n \rightarrow \infty} (1/n) = 0$

Let $a_n = \frac{1}{n}$ and $\epsilon > 0$ be given.

$$|a_n - l| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n}.$$

$$\text{and } |a_n - l| < \epsilon \text{ if } \frac{1}{n} < \epsilon \\ \text{if } n > \frac{1}{\epsilon}.$$

Thus if we take natural no n_1 greater than real no $\frac{1}{\epsilon}$, then we have.

$$|a_n - l| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n \rightarrow 0$$

Explanation For each $\epsilon > 0$, we can always trace n_1 by relation $n > \frac{1}{\epsilon}$ let us see how.

33

13

for $\epsilon = .1$
 $\frac{1}{\epsilon} = \frac{1}{.1} = 10$

our $\frac{1}{n_1}$ will be
greater than 10
because for $a_{10} = \frac{1}{10}$
we have
 $|a_{10} - 0| = |\frac{1}{10} - 0| = .1$

$$a_{11} = \frac{1}{11}$$

$$|a_{11} - 0| = \frac{1}{11} = .09 < \epsilon = .1$$

Thus for $\epsilon = .1$ $n_1 = 11$ &
 $|a_n - 0| < \epsilon \quad \forall n \geq n_1$

for $\epsilon = .01$

$$\frac{1}{\epsilon} = \frac{1}{.01} = 100$$

n_1 will be 101 or greater
and

$$|a_n - 0| < \epsilon = .01 \quad \forall n \geq 101 = n_1$$

We note that for smaller ϵ , the greater n_1

For $\epsilon = .5$

$$\frac{1}{\epsilon} = \frac{1}{.5} = 2$$

n_1 will be 3 or greater
and

$$|a_n - 0| < \epsilon = .5 \quad \forall n \geq 3 = n_1$$

We note that for greater ϵ , the smaller n_1

(b) $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$

let $a_n = \frac{1}{n^2+1}$

$l = 0$ let $\epsilon > 0$ be

given

$$|a_n - l| = \left| \frac{1}{n^2+1} - 0 \right| = \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n}$$

Thus if $\frac{1}{n} < \epsilon$ or $n > \frac{1}{\epsilon}$

we have for n_1 ¹⁴ greater than $\frac{1}{\epsilon}$
 $|a_n - l| < \epsilon$
 $\forall n \geq n_1$

$$\Rightarrow a_n \rightarrow 0$$

$$(c) \quad \lim_{n \rightarrow \infty} \left(\frac{3n+2}{n+1} \right) = 3$$

Let $a_n = \frac{3n+2}{n+1}$ $l = 3$ & $\epsilon > 0$ be given

$$|a_n - l| = \left| \frac{3n+2}{n+1} - 3 \right| = \left| \frac{3n+2-3n-3}{n+1} \right|$$

$$= \left| -\frac{1}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n}$$

Thus $|a_n - l| < \epsilon$ if $\frac{1}{n} < \epsilon$.

So we can take n_1 greater than $\frac{1}{\epsilon}$ for each $\epsilon > 0$ such that

$$|a_n - l| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n \rightarrow l = 3$$

$$(d) \quad \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = 0$$

$$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$|a_n - l| = \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}}$$

$$= \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}}$$

33

If $\frac{1}{\sqrt{n}} < \epsilon$, then $n > \frac{1}{\epsilon^2}$

Thus for each $\epsilon > 0$, if we take n_1 greater than $\frac{1}{\epsilon^2}$, we have.

$$|a_n - l| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n \rightarrow l = 0$$

$$(e) \quad \lim \left(\frac{n^2 - 1}{2n^2 + 3} \right) = \frac{1}{2}$$

$$a_n = \frac{n^2 - 1}{2n^2 + 3} \quad l = \frac{1}{2} \quad \text{let } \epsilon > 0$$

$$|a_n - l| = \left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right|$$

$$= \left| \frac{2n^2 - 2 - 2n^2 - 3}{2(2n^2 + 3)} \right| = \frac{5}{4n^2 + 6} < \frac{5}{4n^2} < \frac{5}{n^2}$$

$$|a_n - l| < \epsilon$$

$$\text{if } \frac{5}{n^2} < \epsilon$$

$$\text{if } \frac{5}{\epsilon} < n^2$$

$$\text{if } n > \sqrt{\frac{5}{\epsilon}}$$

\Rightarrow If we take n_1 greater than $\sqrt{\frac{5}{\epsilon}}$, then

$$|a_n - l| < \epsilon$$

$$\forall n \geq n_1$$

$$\Rightarrow a_n \rightarrow l = \frac{1}{2}$$

(f) If $0 < b < 1$, then $\lim b^n = 0$

for any $a > 0$, we can write.

$$b = \frac{1}{1+a} < 1$$

$$\Rightarrow a = \frac{1}{b} - 1 > 0 \quad \therefore b < 1$$

$$a+1 = \frac{1}{b} \quad \underline{16}$$

Now By Bernoulli's Inequality

$$(1+a)^n \geq 1+na$$

Hence.

$$0 < b^n = \frac{1}{(a+1)^n} \leq \frac{1}{1+na} < \frac{1}{na}$$

$$|a_n - 0| = |b^n - 0| = b^n < \frac{1}{na}$$

$$|a_n - 0| < \epsilon$$

$$\text{if } \frac{1}{na} < \epsilon$$

$$\text{if } \frac{1}{n} < a\epsilon$$

$$\text{if } n > \frac{1}{a\epsilon}$$

\Rightarrow we can take $n_1 > \frac{1}{a\epsilon}$ for each $\epsilon > 0$
such that

$$|a_n - 0| < \epsilon$$

$$\forall n \geq n_1$$

OR

$$|a_n - 0| = b^n$$

$$|a_n - 0| < \epsilon$$

$$\text{if } b^n < \epsilon$$

$$\text{if } n \ln b < \ln \epsilon$$

$$\text{if } n > \frac{\ln \epsilon}{\ln b}$$

$$\because \ln b < 0$$

Thus if we choose $n_1 > \frac{\ln \epsilon}{\ln b}$, we have

$$|a_n - 0| < \epsilon \quad \forall n \geq n_1$$

e.g. if $b = .8$ & if $\epsilon = .01$ we have $n_1 > \frac{\ln .01}{\ln .8}$
 ≈ 20.6377 . Thus $n_1 = 21$ would be
appropriate for $\epsilon = .01$.

17 Uniqueness of Limit

Theorem # A ^{est.} sequence in R can have.
at most one limit

OR

Every Convergent sequence in R has a unique limit

OR

The limit of the sequence, if it exists, is unique.

Proof # Let $\{a_n\}$ be an arbitrary convergent sequence and A, B be two limits of the sequence.

Suppose that $A \neq B$

$$\text{Now } \frac{|A-B|}{2} > 0$$

$\therefore \{a_n\}$ Converges to A

\therefore for $\epsilon = \frac{|A-B|}{2} \exists$ a natural no n_1

such that

$$|a_n - A| < \frac{|A-B|}{2} \quad \longrightarrow \textcircled{1} \quad \forall n > n_1$$

$\therefore \{a_n\}$ Converges to B

\therefore for $\epsilon = \frac{|A-B|}{2} \exists$ a natural no n_2

such that

$$|a_n - B| < \frac{|A-B|}{2} \quad \forall n > n_2 \quad \longrightarrow \textcircled{2}$$

Let $n_3 = \text{Max}\{n_1, n_2\}$
Then

$$\text{and } |a_n - A| < \frac{|A-B|}{2} \quad \forall n \geq n_3 \rightarrow \textcircled{3}$$

$$|a_n - B| < \frac{|A-B|}{2} \quad \forall n \geq n_3 \rightarrow \textcircled{4}$$

$$|A-B| = |A - a_n + a_n - B|$$

$$\leq |A - a_n| + |a_n - B|$$

$$< \frac{|A-B|}{2} + \frac{|A-B|}{2} \quad \forall n \geq n_3$$

$$\Rightarrow |A-B| < |A-B|$$

which is absurd. Hence $A=B$

\Rightarrow limit is unique.

OR

$$\therefore \lim_{n \rightarrow \infty} a_n = A$$

$$\lim_{n \rightarrow \infty} a_n = B$$

\therefore For any $\epsilon > 0 \exists n_1, n_2$ such that

$$|a_n - A| < \epsilon/2 \quad \forall n \geq n_1$$

$$|a_n - B| < \epsilon/2 \quad \forall n \geq n_2$$

$$\text{Let } n_3 = \max(n_1, n_2)$$

Then

$$|a_n - A| < \epsilon/2 \quad \forall n \geq n_3$$

$$|a_n - B| < \epsilon/2 \quad \forall n \geq n_3$$

$$|A-B| = |A - a_n + a_n - B|$$

$$\begin{aligned}
 |A-B| &\leq |A-a_n| + |a_n-B| \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq n_3 \\
 \Rightarrow |A-B| &< \epsilon \quad \forall n \geq n_3
 \end{aligned}$$

$\therefore \epsilon$ is an arbitrary +ve quantity
 $\therefore |A-B|$ is less than every +ve quantity
 however small and so must be zero

Thus $|A-B| = 0$

$$\Rightarrow |A-B| = 0$$

$$\Rightarrow A-B=0 \Rightarrow A=B$$

\Rightarrow Limit of sequence is unique.

Theorem # Let $\{x_n\}$ be a sequence of real numbers and let $x \in \mathbb{R}$. The following statements are equivalent

(a) # $\{x_n\}$ converges to x .

(b) # For every $\epsilon > 0$ \exists a natural no K such that

$$|x_n - x| < \epsilon \quad \forall n \geq K.$$

(c) # For every $\epsilon > 0$, \exists a natural no K s. that

$$x - \epsilon < x_n < x + \epsilon \quad \forall n \geq K.$$

(d) # For every neighbourhood $V_\epsilon(x)$ of x \exists a natural no K such that $\forall n \geq K, x_n \in V_\epsilon(x)$

18
 a (a) $\Rightarrow b \Rightarrow c \Rightarrow \underline{\underline{d}}$
 Let $\{x_n\}$ be convergent to x .
 Then by definition for every $\epsilon > 0$ \exists
 a natural no K such that

$$|x_n - x| < \epsilon \quad \forall n \geq K.$$

$$\Rightarrow x - \epsilon < x_n < x + \epsilon \quad \forall n \geq K.$$

$$\Rightarrow x_n \in (x - \epsilon, x + \epsilon) \quad \forall n \geq K.$$

\Rightarrow for every ϵ -nbhd $V_\epsilon(x)$ \exists a
 natural no K such that

$$x_n \in V_\epsilon(x) \quad \forall n \geq K.$$

Divergent Sequence

(a) A sequence $\{a_n\}$ is said to diverge
 to $+\infty$ if given any +ve real no K
 , however, large, \exists a natural no n_1
 such that

$$a_n > K \quad \forall n \geq n_1$$

and we write

$$\lim_{n \rightarrow \infty} a_n = \infty \text{ or } a_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

(b) A sequence $\{a_n\}$ is said to diverge
 to $-\infty$ if given any +ve real no K ,

however large, \exists a natural no n_1 such that

$$a_n < -K \quad \forall n \geq n_1$$

We write $\lim_{n \rightarrow \infty} a_n = -\infty$ or $a_n \rightarrow -\infty$

Equivalently a sequence $\{a_n\}$ diverges to $-\infty$ if given any -ve real no k \exists a natural no n_1 s.t

$$a_n < k \quad \forall n \geq n_1$$

(c) A sequence $\{a_n\}$ is said to be divergent sequence if it diverges to $+\infty$ or $-\infty$

Examples

(i) The sequences $\{n\}$ & $\{n^2\}$ diverge to ∞

(ii) The sequence $\{-n\}$ & $\{-n^2\}$ diverge to $-\infty$

Oscillatory Sequence

If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence.

Oscillatory sequences

- (a) A bounded sequence which does not converge is said to oscillate finitely.

(b) e.g. $\{(-1)^n\}$

Here $a_n = (-1)^n$ $a_{2n} = (-1)^{2n} = 1$

$$a_{2n+1} = (-1)^{2n+1} = -1$$

$$\lim_{n \rightarrow \infty} a_{2n} = 1$$

$$\lim_{n \rightarrow \infty} a_{2n+1} = -1$$

$\Rightarrow \lim a_n$ does not exist \Rightarrow sequence does not converge. It is bounded because $|a_n| = 1$

Hence sequence oscillate finitely

- (b) An unbounded sequence which does not diverge is said to oscillate infinitely. e.g. $\{(-1)^n n\}$

$$a_n = (-1)^n n$$

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} (2n) = \infty$$

$$\lim_{n \rightarrow \infty} a_{2n+1} = \lim_{n \rightarrow \infty} (-1)^{2n+1} (2n+1) = -\infty$$

Thus sequence does not diverge

Hence this sequence oscillates infinitely

33
23

Null Sequence

A sequence which converges to zero is said to be a null sequence.

e.g. $\{\frac{1}{n}\}$, $\{\frac{1}{n^2}\}$, $\{\frac{1}{2^n}\}$ & $\{\frac{(-1)^{n-1}}{n}\}$ are null sequences

Note A sequence $\{a_n\}$ is called infinitely small if $\lim a_n = 0$ & infinitely large if $\lim a_n = \infty$

Theorem # Let $\{x_n\}$ be a sequence of real numbers and $x \in \mathbb{R}$. If $\{a_n\}$ is a sequence of real nos. with $\lim a_n = 0$ and if for some $c > 0$ and some natural no n_1 , we have

$$|x_n - x| \leq c a_n \quad \forall n \geq n_1$$

Then it follows that $\lim_{n \rightarrow \infty} (x_n) = x$

Proof $\because \lim_{n \rightarrow \infty} a_n = 0$

\therefore For given $\epsilon > 0$ \exists exists a natural no $k(\epsilon/c)$ such that

$$|a_n - 0| < \epsilon/c \quad \forall n \geq k$$

$$\Rightarrow a_n < \epsilon/c \quad \forall n \geq k$$

Let $n_2 = \text{Max}(K, n_1)$

Then

$$|x_n - x| \leq c a_n$$

and $a_n < \epsilon/c$

By ① and ②

$$\forall n \geq n_2 \rightarrow ①$$

$$\forall n \geq n_2 \rightarrow ②$$

$$|x_n - x| \leq c a_n < c(\epsilon/c) = \epsilon \quad \forall n \geq n_2$$

Since ϵ is arbitrary, we have

$$\lim_{n \rightarrow \infty} x_n = x$$

Examples

1) # If $a > 0$, then $\lim_{n \rightarrow \infty} \left(\frac{1}{1+na} \right) = 0$

Sol # $\because a > 0$

$$\therefore 0 < na < 1+na$$

$$\Rightarrow 0 < \frac{1}{1+na} < \frac{1}{na} = \frac{1}{a} \left(\frac{1}{n} \right)$$

Thus $\left| \frac{1}{1+na} - 0 \right| \leq \left(\frac{1}{a} \right) \frac{1}{n} \quad \forall n \in \mathbb{N}$

Since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ and $\frac{1}{a} > 0 \neq n_1 = 1$

from above theorem we have.

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1+na} \right) = 0$$

2) # if $0 < b < 1$, then $\lim_{n \rightarrow \infty} b^n = 0$

Sol $\because 0 < b < 1$

$$\therefore b = \frac{1}{1+a} \stackrel{25}{\Rightarrow} a = \frac{1}{b} - 1$$

$$\text{So } a > 0 \quad (\because b < 1)$$

By Bernoulli's Inequality, we have.

$$(1+a)^n \geq 1+na$$

$$\text{Hence } 0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$$

$$\Rightarrow b^n < \left(\frac{1}{a}\right) \frac{1}{n} \quad \forall n \geq 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \& \frac{1}{a} = c > 0$$

\therefore By above theorem

$$\lim_{n \rightarrow \infty} b^n = 0$$

$$3) \text{ If } c > 0, \text{ then } \lim_{n \rightarrow \infty} c^{1/n} = 1$$

Case I If $c = 1$, then sequence $\{c^{1/n}\}$ is constant sequence $\{1, 1, 1, \dots\}$ which.

Converges to 1.

Case II If $c > 1$, then $c^{1/n} = 1 + d_n$, some $d_n > 0$

By Bernoulli's Inequality

$$c = (1 + d_n)^n \geq 1 + nd_n \quad \forall n \in \mathbb{N}$$

$$\Rightarrow c - 1 \geq nd_n$$

$$\Rightarrow d_n \leq \frac{c-1}{n} \quad \forall n \in \mathbb{N}$$

$$\text{now } |c^{1/n} - 1| = d_n \leq (c-1) \frac{1}{n} \quad \forall n \in \mathbb{N} \quad \begin{cases} c-1 > 0 \\ \therefore c > 1 \end{cases}$$

By above theorem ²⁶

$$\lim_{n \rightarrow \infty} c^{1/n} = 1$$

Case III If $0 < c < 1$, then $c^{1/n} = \frac{1}{1+h_n}$ for some $h_n > 0$. By Bernoulli's Inequality we have

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} < \frac{1}{nh_n}$$

$$\Rightarrow 0 < h_n < \frac{1}{nc} \quad \forall n \in \mathbb{N}$$

$$0 < 1 - c^{1/n} = \frac{h_n}{1+h_n} < h_n < \frac{1}{nc}$$

$$\Rightarrow |c^{1/n} - 1| < \left(\frac{1}{c}\right) \frac{1}{n} \quad \forall n \in \mathbb{N}$$

By above theorem

$$\lim_{n \rightarrow \infty} c^{1/n} = 1$$

4) $\lim_{n \rightarrow \infty} n^{1/n} = 1$

Sol $\therefore n^{1/n} > 1 \quad \forall n > 1$

Let $n^{1/n} = 1 + k_n$ for some $k_n > 0$

$$\Rightarrow n = (1+k_n)^n \quad \forall n > 1$$

$$= 1 + nk_n + \frac{1}{2}n(n-1)k_n^2 + \dots \geq \frac{1}{2}n(n-1)k_n^2$$

$$\Rightarrow n \geq \frac{n(n-1)}{2} k_n^2 + 1$$

$$\begin{aligned}
 & 1 \geq n(n-1)k_n^2 \quad \forall n \geq 1 \\
 & k_n^2 \leq \frac{2}{n} \quad \forall n \geq 1 \\
 & k_n \leq \sqrt{2} \frac{1}{\sqrt{n}} \quad \forall n \geq 1 \\
 & 0 < n^{\frac{1}{n}} - 1 = k_n \leq \sqrt{2} \frac{1}{\sqrt{n}} \quad \forall n \geq 1 \\
 & |n^{\frac{1}{n}} - 1| \leq \sqrt{2} \cdot \frac{1}{\sqrt{n}} \quad \forall n \geq 1 \\
 & \text{By above theorem} \\
 & \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1
 \end{aligned}$$

Theorem # Every Convergent Sequence is bounded.

Proof # Let $\{a_n\}$ be an arbitrary convergent sequence of real number and

$$\lim_{n \rightarrow \infty} a_n = A$$

For $\epsilon = 1$ \exists a natural no n_0 such that

$$|a_n - A| < 1 \quad \forall n \geq n_0 \quad \text{--- (1)}$$

$$|a_n| = |a_n - A + A|$$

$$\leq |a_n - A| + |A|$$

$$< 1 + |A|$$

$$\forall n \geq n_0$$

$$|a_n| < 1 + |A| \quad \underline{28} \quad \forall n \geq m \rightarrow \textcircled{2}$$

Consider the set

$$\{|a_1|, |a_2|, |a_3|, \dots, |a_{m-1}|\} = \{|a_n| : n \leq m-1\}$$

which is finite and has a maximum.

$$\text{Let } M_1 = \max \{|a_n| : n \leq m-1\}$$

$$\text{Then } |a_n| \leq M_1 \quad \forall n \leq m-1 \rightarrow \textcircled{3}$$

$$\text{Let } M = \max \{1 + |A|, M_1\}$$

Then by $\textcircled{2}$

$$|a_n| < 1 + |A| \leq M \quad \forall n \geq m \rightarrow \textcircled{4}$$

$$\text{By } \textcircled{3} \quad |a_n| \leq M_1 \leq M \quad \forall n \leq m-1 \rightarrow \textcircled{5}$$

From $\textcircled{4}$ & $\textcircled{5}$ we have.

$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

$\Rightarrow \{a_n\}$ is bounded.

From $\textcircled{1}$ OR

$$A-1 < a_n < A+1 \quad \forall n \geq m$$

$$\text{Let } k = \min \{a_1, a_2, a_3, \dots, a_{m-1}, A-1\}$$

$$\text{and } k = \max \{a_1, a_2, a_3, \dots, a_{m-1}, A+1\}$$

$$\text{Then } k \leq a_n \quad n \leq m-1 \rightarrow \textcircled{1}$$

$$\text{and } k \leq A-1 < a_n \quad \forall n \geq m \rightarrow \textcircled{2}$$

$$\Rightarrow k \leq a_n \quad \forall n \rightarrow \textcircled{3} \text{ by } \textcircled{1} \text{ \& } \textcircled{2}$$

Also

29

$$\text{and } a_n \leq K \quad \forall n \leq m-1$$

$$a_n < A + 1 \leq K \quad \forall n \geq m.$$

$$\Rightarrow a_n \leq K \quad \forall n \quad \forall n \geq m \rightarrow (5)$$

By (4) & (5)

$$k \leq a_n \leq K \quad \forall n.$$

$\Rightarrow \{a_n\}$ is bounded.

Discussion #1) The intuition for this theorem is quite simple. First by definition of limit all the terms with large index must be close to A . Since the no of terms with small index is finite, and every finite set is bounded, therefore we are able to construct a bound for all terms of the sequence.

(2) # The Converse of the above theorem is not true i.e a bounded sequence is not necessarily Convergent e.g the sequence $\{(-1)^n\}$ is bounded and divergent.

(3) Convergence \longrightarrow Boundedness
Contra-positive if it is

A sequence that is not bounded can never Converge.

This is a useful tool for showing certain sequences do not Converge. The sequence $\{n\}$ diverges because the set of the integers is not bounded.

30

Theorem # If a sequence is unbounded, then it must diverge or it can not converge.

Proof # Suppose that the sequence $\{a_n\}$ is unbounded and let on the contrary it converges and its limit is A .

Then for $\epsilon = 1$ \exists a natural no m such that

$$|a_n - A| < 1 \quad \forall n \geq m.$$

$$|a_n| = |a_n - A + A| \leq |a_n - A| + |A| \leq 1 + |A|$$

if $n < m$, then $\forall n \geq m$.

$$|a_n| \leq \max\{|a_1|, |a_2|, \dots, |a_{m-1}|\}$$

$$\text{Let } M = \max\{|a_1|, |a_2|, \dots, |a_{m-1}|, 1 + |A|\}$$

$$\text{Then } |a_n| \leq M \quad \forall n \in \mathbb{N}.$$

$\Rightarrow \{a_n\}$ is bounded, which is a contradiction.

Hence $\{a_n\}$ is not convergent.

Theorem # If $\{a_n\}$ is a convergent sequence of real numbers such that $a_n \geq 0 \quad \forall n$ and $\lim_{n \rightarrow \infty} a_n = A$, then $A \geq 0$.

Proof # Let on contrary $A < 0$

45

31

Then $-A > 0$

$$\therefore \lim_{n \rightarrow \infty} a_n = A$$

$\therefore \text{For } \epsilon = -A > 0 \exists \text{ a natural no } n_1 \text{ such that}$

$$|a_n - A| < -A = \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow A - \epsilon < a_n < A + \epsilon \quad \forall n \geq n_1$$

In particular, we have

$$a_{n_1} < A + \epsilon = A + (-A) = 0$$

$$\Rightarrow a_{n_1} < 0$$

But this contradicts the hypothesis that $a_n \geq 0 \quad \forall n$. Thus $A \geq 0$

OR

For $\epsilon = \frac{-A}{2} > 0 \exists \text{ a natural no } n_1 \text{ such that}$

$$|a_n - A| < -\frac{A}{2} = \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow a_n - A < -\frac{A}{2} \quad \forall n \geq n_1$$

$$\Rightarrow a_n < \frac{A}{2} < 0 \quad \forall n \geq n_1$$

But by hypothesis $a_n \geq 0 \quad \forall n$.

Hence a contradiction. Thus $A \geq 0$

Note (1) This theorem states that a ~~the~~ sequence of non-negative terms if converges, then converges

to a non-negative limit. 32

(2) If the sequence ultimately becomes non-negative, then if exists, it will have non-negative limit.

Theorem # If a sequence $\{a_n\}$ converges to A , then the sequence $\{|a_n|\}$ converges to $|A|$

Proof # $\because \{a_n\}$ converges to A

\therefore For a given $\epsilon > 0 \exists$ a natural no n_1 such that

$$|a_n - A| < \epsilon \quad \forall n \geq n_1$$

now

$$||a_n| - |A|| \leq |a_n - A| < \epsilon \quad \forall n \geq n_1$$

$\Rightarrow \{ |a_n| \}$ converges to $|A|$

Converse of above is not true $|a_n| = 1$ cgt but $\{(-1)^n\}$ is dgt.

Theorem Let $\{x_n\}$ be a sequence of real nos. that converges to x and suppose that $x_n \geq 0$. Then the sequence $\{\sqrt{x_n}\}$ of the square roots converges and $\lim_{n \rightarrow \infty} (\sqrt{x_n}) = \sqrt{x}$

Proof # $\because x_n \geq 0 \therefore \lim_{n \rightarrow \infty} x_n \geq 0$

So the theorem makes the sense.

Case (i) If $x = 0$, then $x_n \rightarrow 0$ and for given $\epsilon > 0 \exists$ a natural no n_1 such that

$$|x_n - 0| < \epsilon^2 \quad \forall n \geq n_1$$

$$\Rightarrow 0 \leq$$

$\therefore \epsilon$ is a

$$\therefore \sqrt{x_n}$$

Case ii if

$$\sqrt{x_n} - \sqrt{x}$$

$$\therefore \sqrt{x_n} + \sqrt{x}$$

$$\therefore \frac{1}{\sqrt{x_n} + \sqrt{x}} \leq$$

and

$$\sqrt{x_n} - \sqrt{x} =$$

$$\Rightarrow |\sqrt{x_n} - \sqrt{x}| \leq$$

$$\therefore x_n \rightarrow x$$

$$\therefore \text{for } \epsilon \sqrt{x} >$$

such that

$$|x_n -$$

using in ①

$$|\sqrt{x_n} - \sqrt{x}| <$$

34

Theorem # Let $\{a_n\}$ be a sequence.
 $\lim_{n \rightarrow \infty} a_n = 0$ iff $\lim_{n \rightarrow \infty} |a_n| = 0$ i.e. $\{a_n\}$ is
 a null sequence iff $\{|a_n|\}$ is a null sequence.

Proof # Suppose that $\{a_n\}$ is a null
 sequence, then $\lim_{n \rightarrow \infty} a_n = 0$

\therefore Given $\epsilon > 0 \exists$ a true integer n , such
 that

$$|a_n - 0| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow |a_n| < \epsilon \quad \forall n \geq n_1$$

$$\Rightarrow ||a_n| - 0| = ||a_n|| = |a_n| < \epsilon \quad \forall n \geq n_1$$

$\Rightarrow \{|a_n|\}$ is a null sequence.

Now Suppose that $\{|a_n|\}$ is a null sequence
 then $\lim_{n \rightarrow \infty} |a_n| = 0$

\therefore Given $\epsilon > 0 \exists$ a natural no m such that

$$||a_n| - 0| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow ||a_n|| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow |a_n| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow |a_n - 0| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

37

Theorem # ³⁵ If $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence, then $\{a_n b_n\}$ is a null sequence.

Proof # $\because \{b_n\}$ is a bounded sequence
 $\therefore \exists$ a real number M such that
 $|b_n| \leq M \quad \forall n.$

Also $\{a_n\}$ is a null sequence
 \Rightarrow Given $\epsilon > 0, \exists$ a +ve integer m such that

$$|a_n| < \frac{\epsilon}{M} \quad \forall n \geq m$$

$$|a_n b_n - 0| = |a_n| |b_n| < \frac{\epsilon}{M} \cdot M = \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n b_n = 0$$

Theorem # If a sequence $\{a_n\}$ oscillates finitely and $\lim_{n \rightarrow \infty} b_n = 0$, then $\lim_{n \rightarrow \infty} (a_n b_n) = 0$

Proof # $\because \{a_n\}$ oscillates finitely
 $\therefore \{a_n\}$ is bounded.

By above theorem. $\lim_{n \rightarrow \infty} a_n b_n = 0$

Theorem # If $\{a_n\}$ is a null sequence and c is a constant, then $\{c a_n\}$ is a null sequence.

Proof # $\because \{a_n\}$ is a null sequence ³⁶

\Rightarrow Given $\epsilon > 0, \exists$ a true integer m .
Such that

$$|a_n| < \frac{\epsilon}{K+1} \quad \forall n \geq m$$

Now

$$|c a_n| = K |a_n| < \left(\frac{K}{K+1}\right) \epsilon < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} c a_n = 0$$

$\Rightarrow \{c a_n\}$ is a null sequence.

Subsequence

If a new sequence is constructed from the terms of old sequence by picking out terms in any way (but preserving the original order) but in the same order as in original, then new sequence is called a subsequence of the old sequence.

OR

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function with $f(k)$ denoted by n_k . If $\{a_n\}$ is any sequence, then $\{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

Note $a: \mathbb{N} \rightarrow \mathbb{R}$

$f: \mathbb{N} \rightarrow \mathbb{N}$

$$(a \circ f)(k) = a_{f(k)} = a_{n_k}$$

Explanation # 37

Consider the sequence $\{a_n\}$ defined by
 $a_n = \frac{1}{n}$ i.e.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ ①

If we construct a subsequence by crossing out every other term, we get a subsequence.

$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$

Sub-sequence index N \xrightarrow{f} Super sequence index n \xrightarrow{a} term

$k \xrightarrow{f} f(k) = n_k$

1	\xrightarrow{f}	$f(1) = n_1 = 1$	$\longrightarrow a_1 = 1$
2	\xrightarrow{f}	$f(2) = n_2 = 3$	$\longrightarrow a_3 = \frac{1}{3}$
3	\xrightarrow{f}	$f(3) = n_3 = 5$	$\longrightarrow a_5 = \frac{1}{5}$

$k \xrightarrow{f} f(k) = n_k = 2k-1 \longrightarrow a_{2k-1} = \frac{1}{2k-1}$

Thus subsequence $\{a_{n_k}\} = \{a_{2k-1}\} = \left\{\frac{1}{2k-1}\right\}_{k=1}^{\infty}$
or subsequence is $\left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}$

Note (1) we note that n_k is no f index which corresponds to k th term of the ^{sub}sequence.

(2) $n_k \geq k$

38

If we cross-out every odd numbered term we get the subsequence.

Sub-sequence index	Sequence index (n_k)	term a_{n_k}
1	$\longrightarrow n_1 = 2$	$\xrightarrow{a} a_2 = \frac{1}{2}$
2	$\longrightarrow n_2 = 4$	$\xrightarrow{a} a_4 = \frac{1}{4}$
3	$\longrightarrow n_3 = 6$	$\xrightarrow{a} a_6 = \frac{1}{6}$
...
k	$\longrightarrow n_k = 2k$	$\longrightarrow a_{2k} = \frac{1}{2k}$

Thus the subsequence is $\left\{ \frac{1}{2k} \right\}_{k=1}^{\infty} = \left\{ \frac{1}{2n} \right\}_{n=1}^{\infty}$

Examples

The subsequences of the sequence of the integers $\{n\}$ are.

- The sub-sequence of even integers = $\{2n\}$
 $= \{2, 4, 6, \dots\}$
- The sub-sequence of odd integers = $\{2n-1\}_{n=1}^{\infty}$
 $= \{1, 3, 5, \dots\}$
- The subsequence of primes $2, 3, 5, 7, 11, \dots$

Remarks # (1) # The terms of a subsequence occur in the same order in which they occur in the original sequence.

(2) # Every sequence is a subsequence of itself

(3) # The interval in the various terms of a subsequence need not be regular.

(4) # Given a term a_m of a sequence $\{a_n\}$, there is a term of the subsequence following it.

Theorem # If a sequence $\{a_n\}$ converges to A , then every subsequence of $\{a_n\}$ converges to A .

Proof # Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$

$\because \{a_n\}$ converges to A

\therefore Given $\epsilon > 0$, \exists a natural no N_1 such that

$$|a_n - A| < \epsilon \quad \forall n \geq N_1 \rightarrow \textcircled{1}$$

Since n_k is strictly increasing sequence and $n \geq N_1$, therefore $n_n \geq n \geq N_1$. Thus if $k \geq N_1$, then we have $n_k \geq k \geq N_1$ and from $\textcircled{1}$

$$|a_{n_k} - A| < \epsilon \quad \forall n_k \geq N_1$$

$\Rightarrow \{a_{n_k}\}$ converges to A .

Q# Let $a_n = \frac{64}{n}$

Note # 1 # ⁴⁰ The converse of the above theorem is not true i.e. if a subsequence or even an infinitely many subsequences of a given sequence converge, the original sequence may not converge. e.g. let $a_n = (-1)^n$. Then $\{a_n\}$ does not converge. However the two subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ converge to -1 and 1 respectively.

2 # If all subsequences of a sequence $\{a_n\}$ converge to the same limit, only then $\{a_n\}$ converges to that limit.

3 # To prove that a sequence is not convergent it is sufficient to show that two of its subsequences converge to different limit.

In fact $\lim_{n \rightarrow \infty} a_n = A$ iff every subsequence converges to the same limit.

Example

The sequences $\{\frac{1}{2^n}\}$ & $\{\frac{1}{n!}\}$ are subsequences of the convergent sequence $\{\frac{1}{n}\}$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

$$\text{Thus } \lim_{n \rightarrow \infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Theorem # ^(extra) A sequence $\{a_n\}$ converges to a limit A iff the subsequences of even numbered terms and odd numbered terms i.e. $\{a_{2n}\}$ & $\{a_{2n-1}\}$ both converge to A .

Proof # Let $\{a_n\}$ converges to A . Then for

given $\epsilon > 0$, \exists a natural ⁴¹ no N_1 such that
 $|a_n - A| < \epsilon \quad \forall n \geq N_1$

$$\Rightarrow |a_{2n} - A| < \epsilon \quad \forall (2n) \geq N_1$$

$$\text{and } |a_{2n-1} - A| < \epsilon \quad \forall (2n-1) \geq N_1$$

\Rightarrow Subsequences $\{a_{2n}\}$ & $\{a_{2n-1}\}$ Converge to A

Converse Let $\{a_{2n}\}$ & $\{a_{2n-1}\}$ both converge to A. Then for given $\epsilon > 0 \exists$ natural nos N_1 & N_2 such that

$$|a_{2n} - A| < \epsilon \quad \forall 2n \geq N_1$$

$$\text{and } |a_{2n-1} - A| < \epsilon \quad \forall (2n-1) \geq N_2$$

Let $N_3 = \max(N_1, N_2)$, then

$$|a_{2n} - A| < \epsilon \quad \forall (2n) \geq N_3$$

$$\text{and } |a_{2n-1} - A| < \epsilon \quad \forall (2n-1) \geq N_3$$

$$\Rightarrow |a_n - A| < \epsilon \quad \forall n \geq N_3$$

$\Rightarrow \{a_n\}$ Converges to A.

Examples

The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \frac{1}{2^4}, \frac{1}{3^4}, \dots$$

Converges to zero since even numbered and odd numbered terms both converge to 0.

Sequence $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \dots$

diverges because even and odd numbered terms converge to 0 & 1

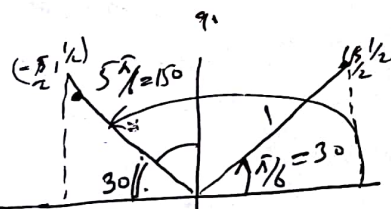
Q# Let a $\frac{64}{h}$

Q# prove that sequence $\{ \sin n \}$ diverges

Sol We use elementary properties of sine function.

We note that

$$\sin \frac{\pi}{6} = \frac{1}{2} = \sin \frac{5\pi}{6}$$



and

$\sin x > \frac{1}{2}$ in interval $(\frac{\pi}{6}, \frac{5\pi}{6}) = I_1$

\therefore The length of the interval $I_1 = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6} > 2$

\therefore There are at least two natural numbers lying inside I_1 . let n_1 be the 1st such number.

Similarly for each $k \in \mathbb{N}$

$$\sin x > \frac{1}{2} \quad \forall x \in \left(\frac{\pi}{6} + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1) \right)$$

$$\text{Let } I_k = \left(\frac{\pi}{6} + 2\pi(k-1), \frac{5\pi}{6} + 2\pi(k-1) \right)$$

\therefore The length of I_k is greater than 2

\therefore There are at least two natural numbers lying inside I_k .

We let n_k be the 1st one. The subsequence.

$\{ \sin n_k \}$ of $\{ \sin n \}$ obtained in this has property

that

$$\sin n_k \in [\frac{1}{2}, 1] \quad \forall n_k$$

Similarly if $k \in \mathbb{N}$ and J_k is the interval

$$J_k = \left(\frac{7\pi}{6} + 2\pi(k-1), \frac{11\pi}{6} + 2\pi(k-1) \right), \text{ then}$$

$$\sin x < -\frac{1}{2}$$

$$\forall x \in J_k$$

42

Theorem # (a) If a sequence $\{a_n\}$ diverges to $+\infty$, then every subsequence of $\{a_n\}$ also diverges to $+\infty$.

(b) If a sequence $\{a_n\}$ diverges to $-\infty$, then every subsequence of $\{a_n\}$ also diverges to $-\infty$.

Proof # Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

$\therefore \{a_n\}$ diverges to $+\infty$

\therefore For every +ve real no K , however large,
 \exists a natural no m such that

$$a_n > K \quad \forall n \geq m.$$

$\therefore \{n_k\}$ is strictly increasing sequence

\therefore For $k_0 \geq m$, we have $n_{k_0} \geq k_0 \geq m$.

and hence for $n_k \geq n_{k_0} \geq m$

$$a_{n_k} > K \quad \forall n_k \geq m.$$

$\Rightarrow \{a_{n_k}\}$ diverges to $+\infty$

(b) Try yourself

Note [1] # The converse of above theorem is not true i.e. if a subsequence of a given sequence diverges to $+\infty$ ($-\infty$), then the sequence need not diverge to $+\infty$ (or $-\infty$) e.g.

$$\text{let } a_n = (-1)^n n = \begin{cases} -n & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

Then sub-sequence $\{a_{2n-1}\}$ diverges to $-\infty$ & the subsequence $\{a_{2n}\}$ diverges to $+\infty$ but the sequence does not diverge to $+\infty$ or $-\infty$ but oscillates.

[2] # If all subsequences diverge to $+\infty$ ($-\infty$) only then the sequence diverges to $+\infty$ ($-\infty$).

Q# Let a_n be a sequence of real numbers

∴ length of J_k is $\frac{1}{k}$ greater than 2.

∴ There are at least two natural numbers lying inside J_k . Let m_k be the 1st natural number lying in J_k .

The subsequence $\{\sin m_k\}$ of $\{\sin n\}$ is such that

$$\sin m_k \in [-1, -\frac{1}{2}] \quad \forall m_k.$$

$$\text{Now } [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$$

Let c be any real number. Then at least one of the subsequences $\{\sin m_k\}$ & $\{\sin m_{k'}\}$ lie entirely outside of $\frac{1}{2}$ -neighbourhood of c i.e. $(c - \frac{1}{2}, c + \frac{1}{2})$. Therefore c can not be a limit of that subsequence. Since $c \in \mathbb{R}$ is an arbitrary, therefore that subsequence is divergent and the sequence is also divergent.

Q# A sequence defined by

$$a_n = (1 - \frac{1}{n}) \sin \frac{n\pi}{2}$$

diverges

Sol We have.

$$a_{2k} = (1 - \frac{1}{2k}) \sin k\pi = (1 - \frac{1}{2k}) \cdot 0 = 0 \quad k=1, 2, \dots$$

$$\begin{aligned} a_{4k+1} &= (1 - \frac{1}{4k+1}) \sin(4k+1)\frac{\pi}{2} \\ &= (1 - \frac{1}{4k+1}) (1) \quad k=1, 2, \dots \end{aligned}$$

We note that subsequences $\{a_{2k}\}$ and $\{a_{4k+1}\}$ converges to 0 & 1

45
Convergence of Constant Sequence

Theorem # A Constant Sequence is cgt.
OR

If $a_n = c \quad \forall n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} a_n = c$$

Proof # Let $\epsilon > 0$, then

$$|a_n - c| = |c - c| = 0 < \epsilon \quad \forall n \geq 1 = n_1$$

$$\Rightarrow a_n \rightarrow c$$

ALGEBRA OF LIMITS #

If $\{a_n\}$ & $\{b_n\}$ are sequences of real nos, we define their sum to be

$$\{a_n + b_n\}$$

addition is performed term by term.

Product by $\{a_n b_n\}$

Difference by $\{a_n - b_n\}$

If $b_n \neq 0 \quad \forall n \in \mathbb{N}$, then division is defined by

$$\frac{\{a_n\}}{\{b_n\}} = \left\{ \frac{a_n}{b_n} \right\}$$

Multiple of sequence $\{a_n\}$ by $c \in \mathbb{R}$ is defined by $\{c a_n\}$

Q. #

Let c be a constant. Then

Theorem # If $\{a_n\}$ & $\{b_n\}$ be sequence of real nos. that converge to A & B respectively and $c \in \mathbb{R}$. Then

- (a) $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$ (a₁) $\lim_{n \rightarrow \infty} c a_n = c A$
 (b) $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
 (c) $\lim_{n \rightarrow \infty} (a_n b_n) = A B$
 (d) If $a_n \neq 0 \forall n$ and $A \neq 0$, then
 $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{A}$
 (e) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ provided $b_n \neq 0$ & $B \neq 0$

Proof # (a₁) $\because \lim_{n \rightarrow \infty} a_n = A$

\therefore Given $\epsilon > 0$ \exists even integer

m such that

$$|a_n - A| < \frac{\epsilon}{|c| + 1} \quad \forall n \geq m$$

$$|c a_n - c A| = |c| |a_n - A|$$

$$< |c| \frac{\epsilon}{|c| + 1} < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} c a_n = c A$$

OR

If $c = 0$, then result is obvious because

$$|c a_n - c A| = 0 < \epsilon \quad \forall n \quad \forall \epsilon > 0$$

Let $c \neq 0$
 $a_n \rightarrow A$

\therefore for given $\epsilon > 0$ \exists a natural no m

Such that

47

$$|a_n - A| < \frac{\epsilon}{|c|}$$

$$|c a_n - c A| = |c| |a_n - A|$$

$$< |c| \frac{\epsilon}{|c|} = \epsilon \quad \forall n \geq m$$

$$\Rightarrow |c a_n - c A| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} c a_n = c A$$

(a) Given $\epsilon > 0$

$$\because \lim_{n \rightarrow \infty} a_n = A \quad \lim_{n \rightarrow \infty} b_n = B$$

$\therefore \exists$ natural nos $m_1 \neq m_2$ such that

$$|a_n - A| < \epsilon/2 \quad \forall n \geq m_1 \rightarrow \textcircled{1}$$

$$|b_n - B| < \epsilon/2 \quad \forall n \geq m_2 \rightarrow \textcircled{2}$$

Let $m = \max(m_1, m_2)$. Then

$$|a_n - A| < \epsilon/2 \quad \rightarrow \textcircled{3}$$

$$|b_n - B| < \epsilon/2 \quad \rightarrow \textcircled{4}$$

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)|$$

$$\leq |a_n - A| + |b_n - B|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_n + b_n) = A + B$$

Q. #

Let a $\frac{64}{48}$ be a constant. Then

$$(b) \quad |(a_n + b_n) - (A - B)| = |(a_n - A) - (b_n - B)| \\ \leq |a_n - A| + |b_n - B| \\ < \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq m$$

$$\Rightarrow |(a_n + b_n) - (A - B)| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_n + b_n) = A - B$$

Note The converse of (a) & (b) not necessarily true i.e. existence $\lim_{n \rightarrow \infty} (a_n + b_n)$ does not necessarily imply that $\lim_{n \rightarrow \infty} a_n$ & $\lim_{n \rightarrow \infty} b_n$ also exist. e.g. let $a_n = n$ $b_n = -n$ both divergent but $a_n + b_n = 0 \quad \forall n$ & $\lim_{n \rightarrow \infty} (a_n + b_n) = 0$
let $a_n = n$ $b_n = n$
 $a_n - b_n = 0$ & $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ but

$\{a_n\}$ & $\{b_n\}$ are divergent

(c) Let $\epsilon > 0$ be given.

$$|a_n \cdot b_n - AB| = |a_n \cdot b_n - b_n \cdot A + b_n A - AB| \\ \leq |a_n \cdot b_n - b_n \cdot A| + |b_n A - AB| \\ = |b_n| |a_n - A| + |A| |b_n - B| \rightarrow \textcircled{1}$$

\therefore The sequence $\{b_n\}$ being cgt is bounded.

$\therefore \exists$ a no M such that
 $|b_n| \leq M$

49

$$\therefore \lim_{n \rightarrow \infty} a_n = A \quad \& \quad \lim_{n \rightarrow \infty} b_n = B$$

$\therefore \exists$ nos $m_1, m_2 \in \mathbb{N}$ such that

$$|a_n - A| < \frac{\epsilon}{2M} \quad \forall n \geq m_1 \rightarrow$$

and $|b_n - B| < \frac{\epsilon}{2(|A|+1)} \quad \forall n \geq m_2$

$\left[\frac{\epsilon}{2(|A|+1)} \right]$ used.
 rather than $\frac{\epsilon}{2|A|}$ to avoid
 from $A=0$

Let $m = \max(m_1, m_2)$. Then.

$$|a_n - A| < \frac{\epsilon}{2M} \quad \rightarrow \textcircled{2}$$

$$|b_n - B| < \frac{\epsilon}{2(|A|+1)} \quad \rightarrow \textcircled{3}$$

From $\textcircled{1}$, $\textcircled{2}$ & $\textcircled{3}$

$$\begin{aligned}
 |a_n \cdot b_n - AB| &\leq |b_n| |a_n - A| + |A| |b_n - B| \\
 &< M \cdot \frac{\epsilon}{2M} + |A| \cdot \frac{\epsilon}{2(|A|+1)} \quad \forall n \geq m \\
 &= \frac{\epsilon}{2} + \left(\frac{|A|}{|A|+1} \right) \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
 &\quad \forall n \geq m
 \end{aligned}$$

$$\Rightarrow |a_n \cdot b_n - AB| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n b_n = AB$$

Q# Let a be a constant. Then

Corollary# Let $\{a_n\}$ be a sequence such that $\lim_{n \rightarrow \infty} a_n = l$ and k be any +ve integer. Then $\lim_{n \rightarrow \infty} a_n^k = l^k$.

Proof# By induction on k

For $k=2$

$$\lim_{n \rightarrow \infty} a_n^2 = \lim_{n \rightarrow \infty} a_n \cdot a_n = l \cdot l = l^2$$

\Rightarrow It is true for $k=2$

Let it be true for $k=p$ i.e

$$\lim_{n \rightarrow \infty} a_n^p = l^p \quad p \geq 2$$

$$\Rightarrow l \cdot \lim_{n \rightarrow \infty} a_n^p = l \cdot l^p$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} a_n^p = l^{p+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_n \cdot a_n^p) = l^{p+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n^{p+1} = l^{p+1}$$

\Rightarrow It is true for $k=p+1$

Thus it is true for all +ve integral values of k and

$$\lim_{n \rightarrow \infty} a_n^k = l^k$$

Note the converse ⁵¹ of the above theorem ^{is} is not necessarily true i.e. existence. $\lim_{n \rightarrow \infty} (a_n b_n)$ does not necessarily imply that the two limits $\lim_{n \rightarrow \infty} a_n$ & $\lim_{n \rightarrow \infty} b_n$ also exist. e.g.

let $a_n = b_n = (-1)^n$, then both limits $\lim_{n \rightarrow \infty} a_n$ & $\lim_{n \rightarrow \infty} b_n$ do not exist. But

$a_n b_n = (-1)^n (-1)^n = (-1)^{2n} = 1 \quad \forall n$
So that $\lim_{n \rightarrow \infty} (a_n b_n) = 1$ exists.

(d) Fix $\epsilon > 0$.

$$\left| \frac{1}{a_n} - \frac{1}{A} \right| = \left| \frac{A - a_n}{a_n A} \right| = \frac{|a_n - A|}{|a_n| |A|} \quad \text{--- (1)}$$

$$\therefore \lim_{n \rightarrow \infty} a_n = A \quad \therefore \lim_{n \rightarrow \infty} |a_n| = |A|$$

$$\text{Taking } \epsilon = \frac{|A|}{2}$$

\exists a true integer m_1 s. that

$$||a_n| - |A|| < \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow |A| - \frac{|A|}{2} < |a_n| < |A| + \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\frac{|A|}{2} < |a_n| \quad \forall n \geq m_1$$

Q. # Let a ⁵⁴ 64 be a constant. Then

$$|a_n - A| < \frac{|A|}{2} \quad \text{53} \quad \forall n \geq m_1$$

Now

$$||a_n| - |A|| \leq |a_n - A| < \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow -\frac{|A|}{2} \leq -|a_n - A| \leq |a_n| - |A| \quad \forall n \geq m_1$$

$$\leq |a_n| - |A|$$

$$\Rightarrow -\frac{|A|}{2} \leq |a_n| - |A|$$

$$\Rightarrow \frac{|A|}{2} \leq |a_n|$$

$$\Rightarrow \frac{1}{|a_n|} \leq \frac{2}{|A|} \quad \forall n \geq m_1$$

$$\text{OR}$$

$$|a_n - A| < \frac{|A|}{2} \quad \forall n \geq m_1$$

$$\Rightarrow |a_n - A| + |a_n| < \frac{|A|}{2} + |a_n| \quad \forall n \geq m_1$$

→ (a)

$$|A| = |A - a_n + a_n| \leq |A - a_n| + |a_n|$$

$$\Rightarrow |A| \leq |A - a_n| + |a_n| < \frac{|A|}{2} + |a_n| \quad \forall n \geq m_1$$

$$\Rightarrow \frac{|A|}{2} < |a_n|$$

$$\Rightarrow \frac{1}{|a_n|} < \frac{2}{|A|} \quad \forall n \geq m_1$$

$$\frac{2}{|A|} > \frac{1}{|a_n|} \quad \forall n \geq m_1$$

$$\frac{1}{|a_n|} < \frac{2}{|A|} \quad \forall n \geq m_1$$

Also $\lim_{n \rightarrow \infty} a_n = A$, therefore \exists natural no m_2 s. that

$$|a_n - A| < \frac{|A|^2 \epsilon}{2} \quad \forall n \geq m_2$$

Let $m = \max(m_1, m_2)$. Then.

$$\frac{1}{|a_n|} < \frac{2}{|A|} \quad \forall n \geq m \rightarrow \textcircled{2}$$

$$\text{and } |a_n - A| < \frac{|A|^2 \epsilon}{2} \quad \forall n \geq m \rightarrow \textcircled{3}$$

From $\textcircled{1}$ $\textcircled{2}$ & $\textcircled{3}$

$$\left| \frac{1}{a_n} - \frac{1}{A} \right| = \frac{|a_n - A|}{|a_n| |A|}$$

$$< \frac{2}{|A|} \cdot \frac{1}{|A|} \cdot \frac{|A|^2 \epsilon}{2} = \epsilon \quad \forall n \geq m.$$

$$\Rightarrow \left| \frac{1}{a_n} - \frac{1}{A} \right| < \epsilon \quad \forall n \geq m.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{A}$$

OR

$$\therefore \lim_{n \rightarrow \infty} a_n = A$$

\therefore For $\epsilon = \frac{|A|}{2} > 0 \quad \exists m_1 \in \mathbb{N}$ s. that

Remark

terms

essentially

from zero

eventually

limit of

$\epsilon) \#$ for

$$\left| \frac{a_n}{b_n} - \frac{A}{B} \right| >$$

$$= \frac{|B| |a_n - \frac{A}{B} b_n|}{|b_n|}$$

$$= \frac{|B| |a_n b_n - A|}{|b_n|}$$

$$= \frac{|B| |a_n b_n - A|}{|b_n|}$$

$$= \frac{|B| |a_n b_n - A|}{|b_n|}$$

$$= \frac{|B| |a_n b_n - A|}{|b_n|}$$

$$\lim_{n \rightarrow \infty} \frac{|B| |a_n b_n - A|}{|b_n|}$$

$$\Rightarrow a$$

$$|b_n - B|$$

$$\frac{|B|}{2} >$$

$$|b_n|$$

$$|b_n|$$

$$\frac{0}{2}$$

$$\begin{aligned}
 & | |b_n| - |B| | \leq \frac{54+1}{2} = \frac{55}{2} \quad \forall n \geq m_1 \\
 \Rightarrow & |B| - \frac{|B|}{2} < |b_n| < |B| + \frac{|B|}{2} \quad " " \\
 \Rightarrow & |b_n| > \frac{|B|}{2} \quad \forall n \geq m_1 \\
 \Rightarrow & \frac{1}{|b_n|} < \frac{2}{|B|} \quad \forall n \geq m_1 \quad \rightarrow (2)
 \end{aligned}$$

Also $\lim_{n \rightarrow \infty} a_n = A$ & $\lim_{n \rightarrow \infty} b_n = B$

$\Rightarrow \exists$ natural nos m_2, m_3 such that

$$|a_n - A| < \frac{|B|\epsilon}{4} \quad \forall n \geq m_2 \rightarrow (3)$$

$$|b_n - B| < \frac{|B|^2 \epsilon}{4(|A|+1)} \quad \forall n \geq m_3 \rightarrow (4)$$

Let $m = \max(m_1, m_2, m_3)$. Then.

$$\frac{1}{|b_n|} < \frac{2}{|B|} \quad \forall n \geq m \rightarrow (5)$$

$$|a_n - A| < \frac{|B|\epsilon}{4} \quad " " \rightarrow (6)$$

$$|b_n - B| < \frac{|B|^2 \epsilon}{4(|A|+1)} \quad " " \rightarrow (7)$$

From (1), (5), (6) & (7)

$$\begin{aligned}
 \left| \frac{a_n}{b_n} - \frac{A}{B} \right| & \leq \frac{|a_n - A|}{|b_n|} + \frac{|A| |b_n - B|}{|b_n| |B|} \\
 & < \frac{2}{|B|} \cdot \frac{|B|\epsilon}{4} + \frac{2}{|B|} \cdot \frac{|A|}{|B|} \cdot \frac{|B|^2 \epsilon}{4(|A|+1)} \quad \forall n \geq m
 \end{aligned}$$

Q# Let a be a constant. Then

$$\underline{55} + 1 = \underline{56}$$

$$= \frac{\epsilon}{2} + \frac{|A|}{|A|+1} \cdot \frac{\epsilon}{2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq m$$

$$\Rightarrow \left| \frac{a_n}{b_n} - \frac{A}{B} \right| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$$

Note # The converse of the above theorem is not necessarily true i.e. the existence of

$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right)$ does not necessarily imply that the two limits $\lim_{n \rightarrow \infty} a_n$ & $\lim_{n \rightarrow \infty} b_n$ also exist.

e.g. $a_n = b_n = n$, then $\lim_{n \rightarrow \infty} a_n$, $\lim_{n \rightarrow \infty} b_n$ does not exist. But $\frac{a_n}{b_n} = \frac{1}{1} = 1$ & $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$

Theorem # If $\lim_{n \rightarrow \infty} a_n = A$ & $A \neq 0$, then

\exists a +ve integer k and natural no m such that $|a_n| > k > 0$ i.e. sequence is eventually bounded away from zero

Proof # $\because A \neq 0 \quad \therefore |A| > 0$

Also $\lim_{n \rightarrow \infty} a_n = A$

\Rightarrow For $\epsilon = \frac{|A|}{2}$, \exists natural no. m s.t.

$$|a_n - A| < \frac{|A|}{2} \quad \forall n \geq m$$

$$\begin{aligned} \text{Now } |A| &= |A - \overset{56+1=57}{a_n} + a_n| \\ &\leq |a_n| + |a_n - A| \\ &< |a_n| + \frac{|A|}{2} \quad \forall n \geq m \end{aligned}$$

$$\Rightarrow \frac{|A|}{2} < |a_n| \quad \forall n \geq m$$

$$\Rightarrow |a_n| > \frac{|A|}{2} > 0 \quad \forall n \geq m$$

$$\Rightarrow |a_n| > \frac{|A|}{2} = k \quad \forall n \geq m$$

Theorem # (Limit is order preserving on convergent sequences)

If $\lim_{n \rightarrow \infty} a_n = A$ & $\lim_{n \rightarrow \infty} b_n = B$, $a_n \leq b_n \quad \forall n$

Then $A \leq B$ or $\lim a_n \leq \lim b_n$.

Proof # Let $c_n = b_n - a_n \quad \forall n$.

Then $c_n \geq 0 \quad \forall n \quad \because b_n \geq a_n \quad \forall n$

$$\Rightarrow \lim_{n \rightarrow \infty} c_n \geq 0 \quad \forall$$

$$\Rightarrow \lim_{n \rightarrow \infty} (b_n - a_n) \geq 0$$

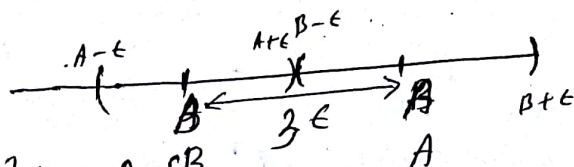
$$\Rightarrow \lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} a_n \geq 0$$

$$\Rightarrow B - A \geq 0$$

$$\Rightarrow A \leq B \quad \text{proved}$$

Q# Let a ⁶⁴ be a constant. Then
 $\underline{57} + 1 = \underline{58}$
 on Contrary let OR $A > B$ and.

$A - B = 3\epsilon$
 So that nbhds $]B - \epsilon, B + \epsilon[\cap]A - \epsilon, A + \epsilon[$



of B and A are disjoint

$\therefore a_n \rightarrow A \neq b_n \rightarrow B$
 $\therefore \exists$ natural no $m_1 \neq m_2$ such that

$$A - \epsilon < a_n < A + \epsilon \quad \forall n \geq m_1$$

$$B - \epsilon < b_n < B + \epsilon \quad \forall n \geq m_2$$

Let $m = \max(m_1, m_2)$

Then $a_n \in]A - \epsilon, A + \epsilon[\quad \forall n \geq m$

$b_n \in]B - \epsilon, B + \epsilon[\quad \forall n \geq m$

$\Rightarrow b_n < a_n \quad \forall n \geq m$

which contradicts the fact that

$$a_n \leq b_n \quad \forall n$$

Hence our supposition is wrong and.

$$A \leq B$$

Corollary If $a_n \leq k \quad \forall n$ & $\lim_{n \rightarrow \infty} a_n = A$, then $A \leq k$.

Let $b_n = k - a_n$. Then $b_n \geq 0 \quad \forall n$.

$$\Rightarrow \lim b_n \geq 0 \Rightarrow \lim (k - a_n) \geq 0$$

$$\Rightarrow k - A \geq 0 \Rightarrow k \geq A.$$

Theorem # If $\{a_n\}$ be sequence such that
 $a \leq a_n \leq b \quad \forall n \in \mathbb{N}$

Then

$$a \leq \lim_{n \rightarrow \infty} a_n \leq b$$

Proof #

Consider sequence $\{b, b, b, \dots\} = \{c_n\}$

Then $a_n \leq c_n \quad \forall n$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} c_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n \leq b \rightarrow \textcircled{1}$$

Consider sequence $\{d_n\} = \{a, a, a, \dots\}$

Then $d_n \leq a_n \quad \forall n$

$$\Rightarrow \lim_{n \rightarrow \infty} d_n \leq \lim_{n \rightarrow \infty} a_n$$

$$a \leq \lim_{n \rightarrow \infty} a_n \rightarrow \textcircled{2}$$

By $\textcircled{1}$ & $\textcircled{2}$

$$a \leq \lim_{n \rightarrow \infty} a_n \leq b$$

Theorem # (Squeeze Theorem, Sandwich.
 (old squeeze play) Theorem) Trapped sequences

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be the sequences
 such that

$$a_n \leq b_n \leq c_n \quad \forall n$$

$$(1) \text{ and } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = l$$

$$\text{Then } \lim_{n \rightarrow \infty} b_n = l$$

(ii) For some ⁶⁴60 integer p ,
 $a_n \leq b_n \leq c_n \quad \forall n \geq p$
 then $\lim_{n \rightarrow \infty} b_n = l$

Proof # Let $\epsilon > 0$ be given.

$$\because \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = l.$$

\therefore \exists even integers m_1 & m_2 such that
 $\forall n \geq m_1$

$$|a_n - l| < \epsilon$$

$$\forall n \geq m_2$$

$$|c_n - l| < \epsilon$$

Let $m = \max(m_1, m_2)$. Then.

$$|a_n - l| < \epsilon$$

$$\forall n \geq m$$

$$|c_n - l| < \epsilon$$

$$\forall n \geq m$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m.$$

$$l - \epsilon < c_n < l + \epsilon \quad \forall n \geq m.$$

Thus

$$l - \epsilon < a_n \leq b_n \leq c_n < l + \epsilon \quad \forall n \geq m.$$

$$\Rightarrow l - \epsilon < b_n < l + \epsilon \quad \forall n \geq m.$$

$$\Rightarrow |b_n - l| < \epsilon \quad \forall n \geq m.$$

$$\Rightarrow \lim_{n \rightarrow \infty} b_n = l.$$

When

6.1

Then we take $a_n \leq b_n \leq c_n \quad \forall n \geq p$
So that $m = \max(m_1, m_2, p)$

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

$$|c_n - l| < \epsilon \quad \forall n \geq m$$

$$\& \quad a_n \leq b_n \leq c_n \quad \forall n \geq m$$

$$\Rightarrow l - \epsilon < a_n \leq b_n \leq c_n < l + \epsilon \quad \forall n \geq m$$

$$\Rightarrow |b_n - l| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} b_n = l$$

Applications

Q.1 # $\lim_{n \rightarrow \infty} \left(\frac{\sin n}{n} \right) = 0$

We can not apply $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = A/B$

because sequence $\{n\}$ is not convergent
However, we have

$$-1 \leq \sin n \leq 1 \quad \forall n$$

$$\Rightarrow -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \right) = 0$$

\therefore By Squeeze theorem.

Q# Let $a > 1$

Q# Find $\lim_{n \rightarrow \infty} a^{1/n}$, where a is a fixed real number.

Sol# Case I# If $a > 1$, then

$$a^{1/n} > 1$$

$$\text{Let } b_n = a^{1/n} - 1$$

$$a^{1/n} = b_n + 1 \quad \text{where } b_n > 0$$

$$\Rightarrow a = (1 + b_n)^n > 1 + n b_n \quad \text{Bernoulli's inequality}$$

$$\Rightarrow a - 1 > n b_n$$

$$\Rightarrow 0 \leq b_n \leq \frac{a-1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a-1}{n} = (a-1) \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

\therefore By squeeze play

$$\lim_{n \rightarrow \infty} b_n = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} a^{1/n} - 1 = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} a^{1/n} = 1$$

Case#2# $0 < a < 1$, then $1/a > 1$
and $\lim_{n \rightarrow \infty} a^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{(1/a)^{1/n}} = \frac{1}{1} = 1$

$$\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$$

Q# (a) $\lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0$

Sol

$$\therefore -1 \leq \cos n \leq 1$$

$$\therefore -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

\Rightarrow By squeeze play

$$\lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0$$

or $0 \leq \left| \frac{\cos n}{n} \right| \leq \frac{1}{n}$

$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\cos n}{n} \right| = 0$ by squeeze play

and hence $\lim_{n \rightarrow \infty} \frac{\cos n}{n} = 0$

(b) $(-1)^n \frac{1}{n} \rightarrow 0$

$$\therefore 0 \leq \left| (-1)^n \frac{1}{n} \right| \leq \frac{1}{n}$$

$\Rightarrow \lim_{n \rightarrow \infty} (-1)^n \frac{1}{n} = 0$ by squeeze play

(c) $\frac{\cos^2 n}{3^n} \rightarrow 0$ because

$$0 \leq \frac{\cos^2 n}{3^n} < \frac{1}{3^n}$$

$\Rightarrow \lim_{n \rightarrow \infty} \frac{\cos^2 n}{3^n} = 0$ by squeeze play

$\therefore 0 = 0$

49

Q. # Let a be a constant. Then.
prove that

(a) $a^n \rightarrow 0$ if $0 < a < 1$.

(b) $a^n \rightarrow \infty$ if $a > 1$

Sol # $\because 0 < a < 1$

$$\Rightarrow \frac{1}{a} > 1$$

$$\text{Let } \frac{1}{a} = 1 + p \quad p > 0$$

$$\frac{1}{a^n} = (1+p)^n = 1 + np + (\text{+ve terms}) > np$$

$$\frac{1}{a^n} > np$$

$$\Rightarrow 0 < a^n < \frac{1}{np} = \frac{1}{p} \cdot \frac{1}{n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{pn} = 0$$

$$\therefore \text{By Squeeze play } \lim_{n \rightarrow \infty} a^n = 0$$

(b) If $a > 1$, then

$$a = 1 + b \quad \text{where } b > 0$$

$$a^n = (1+b)^n = 1 + nb + (\text{+ve terms}) > 1 + nb \rightarrow \infty$$

$$\Rightarrow a^n \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

65

Ratio Test For Convergence

For certain types of sequences, the following result provide quick & easy ratio test for convergence.

Theorem # If $\{a_n\}$ be a sequence such that $a_n \neq 0$, be a sequence of real terms and

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l \quad \text{where } l < 1$$

then $\lim_{n \rightarrow \infty} a_n = 0$

Proof # $\because a_n > 0$
 $\therefore \lim_{n \rightarrow \infty} a_n > 0 \quad \& \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 0$

$$\Rightarrow l > 0$$

Let ϵ be a number such that

$$l < \epsilon < 1 \quad \text{and let } \epsilon = \epsilon - l > 0$$

Then \exists a natural no m such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon \quad \forall n > m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < l + \epsilon = l + \epsilon - l = \epsilon \quad \forall n > m$$

$$\Rightarrow a_{n+1} < \epsilon a_n \quad \rightarrow \textcircled{A} \quad \forall n > m$$

putting $n = m, m+1, m+2, \dots, n-1$

we $a_{m+1} < \epsilon a_m \quad \rightarrow \textcircled{1}$

$$a_{m+2} < \epsilon a_{m+1} \quad \rightarrow \textcircled{2}$$

m
 $n-m$

$$a_{m+3} < r a_{m+2}$$

$$a_{n-1} < r a_{n-2}$$

$$a_n < r a_{n-1}$$

Multiplying all above inequalities we get

$$a_{m+1} a_{m+2} a_{m+3} \dots a_{n-1} a_n < r^{n-m} a_m a_{m+1} a_{m+2} \dots a_{n-1}$$

$$\Rightarrow a_n < a_m r^{n-m}$$

$$\Rightarrow \frac{a_n}{a_m} < r^{n-m}$$

$$a_n < \frac{a_m r^n}{r^m} \rightarrow \textcircled{B} \quad \forall n \geq m$$

$$\text{But } 0 < r < 1 \quad \therefore r^n \rightarrow 0$$

$$\text{Hence } \lim_{n \rightarrow \infty} a_n = 0$$

$$\therefore r^n \rightarrow 0 \quad \text{OR} \quad \therefore \text{Given } \epsilon > 0 \quad \exists$$

a natural no p such that

$$|r^n - 0| < \frac{\epsilon}{a_m} \quad \forall n \geq p \rightarrow \textcircled{B}$$

Let $m_1 \geq \max(m, p)$, then \textcircled{A} & \textcircled{B} both holds for $n \geq m_1$ and

$$\left\{ \begin{array}{l} m, m+1, \dots, n-1 \\ a_n \geq m+(n-1) \\ n-1 \geq m+(n-1) \\ n-m \geq N \end{array} \right.$$

67

$$a_n < \frac{a_m \epsilon^n}{\epsilon^m} \rightarrow \textcircled{C} \forall n \geq m,$$

$$\} \epsilon^n < \frac{\epsilon^m \epsilon}{a_m} \rightarrow \textcircled{D} \forall n \geq m,$$

$$\Rightarrow a_n < \frac{a_m}{\epsilon^m} \times \frac{\epsilon^m \epsilon}{a_m} = \epsilon \quad \forall n \geq m,$$

$$\Rightarrow \because a_n > 0$$

$$\Rightarrow |a_n - 0| < \epsilon \quad \forall n \geq m,$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

OR

From \textcircled{A}

$$a_{n+1} < \epsilon a_n$$

putting $n-1, n-2, \dots, m-1$

$$a_n < \epsilon a_{n-1}$$

$$a_{n-1} < \epsilon a_{n-2}$$

$$a_{n-2} < \epsilon a_{n-3}$$

$$a_{n-3} < \epsilon a_{n-4}$$

$$\vdots$$

$$a_{m-1} < \epsilon a_{m-2}$$

$$a_m < \epsilon a_{m-1} \quad \text{for } n=m-1$$

$$\cancel{a_{m+1} < \epsilon a_m.}$$

Xing all above equations

49

Theorem # If $\{a_n\}$ be a sequence such that
 $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l > 1$, then
 $\lim_{n \rightarrow \infty} a_n = \infty$

Proof # $\because l > 1 \therefore l-1 > 0$
 we can choose a pos number ϵ such that
 $0 < \epsilon < l-1$ or $l-\epsilon > 1$

$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l \therefore \exists$ a even integer
 m such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon \quad \forall n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > l - \epsilon = k \rightarrow \textcircled{A} \quad \forall n \geq m$$

putting $n = m, m+1, \dots, n-1$ in \textcircled{A}

$$a_{m+1} > k a_m$$

$$a_{m+2} > k a_{m+1}$$

$$a_{m+3} > k a_{m+2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{n-2} > k a_{n-3}$$

$$a_{n-1} > k a_{n-2}$$

$$a_n > k a_{n-1}$$

69

Multiplying all above inequalities

$$a_{m+1} a_{m+2} a_{m+3} \dots a_{n-2} a_{n-1} a_n$$

$$> \epsilon^m a_m a_{m+1} a_{m+2} \dots a_{n-3} a_{n-2} \dots a_{n-1}$$

$$\Rightarrow a_n > \epsilon^{n-m} a_m$$

$$\Rightarrow a_n > \frac{a_m}{\epsilon^m} \epsilon^n \quad \forall n \geq m$$

$$\Rightarrow |a_n| > \frac{|a_m|}{\epsilon^m} \epsilon^n \quad \forall n \geq m$$

$$\therefore \epsilon < 1 \quad \therefore \epsilon^n \rightarrow 0$$

$$\therefore \lim_{n \rightarrow \infty} a_n = 0$$

Q# prove that for any real number x

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

Sol let $a_n = \frac{x^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{x}{n+1} = 0 < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

Q: Apply the above 1st theorem, where a, b satisfy $0 < a < 1, b > 1$

Sol # (a) $\{a^n\}$ (b) $\{\frac{b^n}{2^n}\}$
(c) $\{\frac{n}{b^n}\}$ (d) $\{\frac{2^{3n}}{3^{2n}}\}$

Sol (a)

$$a_n = a^n \quad 0 < a < 1$$

$$a_{n+1} = a^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{a^n} = a < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} a^n = 0$$

(b) let $a_n = \frac{b^n}{2^n}$

$$a_{n+1} = \frac{b^{n+1}}{2^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{2^{n+1}} \times \frac{2^n}{b^n} = \frac{b}{2}$$

Q# Discuss the convergence of following sequences where a, b are such that $0 < a < 1, b > 1$

(a) $\{n^2 a^n\}$

(b) $\{\frac{b^n}{n^2}\}$

(c) $\{\frac{b^n}{n!}\}$

(d) $\frac{n!}{n^n}$

Sol (a) let $a_n = n^2 a^n$

$$a_{n+1} = (n+1)^2 a^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 a^{n+1}}{n^2 a^n} = \left(\frac{n+1}{n}\right)^2 a$$

$$= \left(1 + \frac{1}{n}\right)^2 a$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 a$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

$n!$ eventually increases more rapidly than a^n . This result is very useful.

(b) let $a_n = \frac{b^n}{n^2}$

$$a_{n+1} = \frac{b^{n+1}}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{(n+1)^2} \times \frac{n^2}{b^n} = \frac{n^2}{(n+1)^2} b$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^2} b$$

$$= b > 1$$

$\Rightarrow \{a_n\}$ is dg^+

(c) let $a_n = \frac{n}{b^n}$ $a_{n+1} = \frac{n+1}{b^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{b^{n+1}} \times \frac{b^n}{n} = \frac{1}{b} \left(1 + \frac{1}{n}\right)$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{b} \left(1 + \frac{1}{n}\right) = \frac{1}{b} < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

(d) let $a_n = \frac{n!}{n^n}$

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{n^n}{(n+1)^n}$$

$$\begin{aligned}
 &= \left(\frac{n}{n+1}\right)^n \quad \frac{n}{n+1} < 1 \\
 \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \\
 &= \frac{1}{e} < 1 \\
 \Rightarrow \lim_{n \rightarrow \infty} a_n &= 0
 \end{aligned}$$

Monotone Sequences

(a) Monotone Increasing or Non-decreasing

A sequence $\{a_n\}$ is called monotone (moving in one direction) increasing if

$$a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$$

and is called strictly increasing if

$$a_n < a_{n+1} \quad \forall n \in \mathbb{N}$$

(b) Monoton Decreasing or Non-Increasing Sequence

A sequence $\{a_n\}$ is called non-increasing or monotone decreasing if

$$a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$$

and is called strictly decreasing if

$$a_n > a_{n+1} \quad \forall n \in \mathbb{N}$$

73

(c) # Monotonic Sequence #

A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing

(d) Strictly Monotonic Sequence.

A sequence is said to be strictly monotonic if it is either strictly monotonically increasing or strictly monotonically decreasing

Testing of Monotonicity of a Sequence.

There are several methods of testing whether a sequence $\{a_n\}$ monotone or not

(a) Difference b/w Successive Terms #

Difference

$$a_{n+1} - a_n > 0$$

$$a_{n+1} - a_n < 0$$

$$a_{n+1} - a_n \geq 0$$

$$a_{n+1} - a_n \leq 0$$

Classification.

strictly increasing

,, ,, decreasing

Non-decreasing

Non-increasing

(b) By Ratio of Successive Terms

Ratio

$$\frac{a_{n+1}}{a_n} > 1$$

Classification

increasing

$$\frac{a_{n+1}}{a_n} < 1$$

Decreasing

$$\frac{a_{n+1}}{a_n} \geq 1$$

Non-decreasing

$$\frac{a_{n+1}}{a_n} \leq 1$$

Non-increasing

(c) If $f'(n) > f(n)$ and f is differentiable
Then.

Derivative

$$f'(n) > 0$$

Classification

increasing

$$f'(n) < 0$$

decreasing

$$f'(n) \geq 0$$

Non-decreasing

$$f'(n) \leq 0$$

Non-increasing

(d) Induction use induction on n .

Remarks # If $\{a_n\}$ is increasing, then it is bounded below by a_1 and will be bounded if it is bounded above. If $\{a_n\}$ is decreasing, it is bounded above by a_1 and will be bounded if

it is bounded below. 75

(2) Monotonicity is very useful because it prevents the terms of a sequence from oscillating.

Eventually Monotone or Ultimately Monotone Sequence.

A sequence is eventually or ultimately monotone if \exists an ^{int} integer m such that the sequence is monotone for all $n \geq m$. i.e. sequence is monotone from some term onward.

The Completeness Property of \mathbb{R}

Every non-empty set of real numbers that has an upper bound (is bounded above) also has a Supremum in \mathbb{R} . It is also called the least upper bound property of \mathbb{R} .

Theorem # A monotone sequence of real nos. is convergent iff it is bounded. Further.

(a) If $\{a_n\}$ is bounded monotone increasing sequence, then it converges to its Supremum i.e.

$$\lim_{n \rightarrow \infty} a_n = \sup \{a_n : n \in \mathbb{N}\} = \sup a_n$$

49

(b) If $\{a_n\}$ is monotone bounded below, then it converges to its infimum i.e.

$$\lim_{n \rightarrow \infty} a_n = \inf \{a_n : n \in \mathbb{N}\} \\ = \inf_n a_n$$

Proof

Necessary Condition #

Let $\{a_n\}$ be monotone convergent sequence. Then we have already proved that every cgt sequence is bound

Converse

Conversely let $\{a_n\}$ a bounded monotone sequence. Then \exists nos m & M such that

$$m \leq a_n \leq M \quad \forall n$$

Then $\{a_n\}$ is either increasing or decreasing

(a) Let $\{a_n\}$ be bounded increasing sequence.

Then the range set $S = \{a_n : n \in \mathbb{N}\}$ is bounded above and by least upper bound axiom of \mathbb{R} S has l.u.b exists in \mathbb{R} .

$$\text{Let } L = \sup \{a_n : n \in \mathbb{N}\}$$

Given any $\epsilon > 0$, $L - \epsilon$ is not an upper bound of $\{a_n\}$ and there is at least one term a_m of $\{a_n\}$ greater than $L - \epsilon$ i.e.

$$L - \epsilon < a_m \quad \rightarrow \textcircled{1}$$

otherwise $L - \epsilon$ will be an upper bound.

Since $\{a_n\}$ is monotonically increasing sequence

$$a_m \leq a_{m+1} \leq a_{m+2} \leq \dots \rightarrow \textcircled{2}$$

By $\textcircled{1}$ & $\textcircled{2}$

$$L - \epsilon < a_n \quad \forall n \geq m \rightarrow \textcircled{3}$$

Also L is least upper bound of sequence

$$\therefore a_n \leq L \quad \forall n \in \mathbb{N} \rightarrow \textcircled{4}$$

$$\Rightarrow L - a_n \geq 0 \quad \forall n \rightarrow \textcircled{4}$$

From $\textcircled{3}$ & $\textcircled{4}$

$$0 \leq L - a_n < \epsilon \quad \forall n \geq m$$

$$\Rightarrow |L - a_n| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = L.$$

OR

Since L is supremum, therefore

$$a_n \leq L < L + \epsilon \quad \forall n \in \mathbb{N}$$

$$\Rightarrow a_n < L + \epsilon \quad \forall n \geq m \rightarrow \textcircled{5}$$

By $\textcircled{3}$ & $\textcircled{5}$

$$L - \epsilon < a_n < L + \epsilon \quad \forall n \geq m$$

$$\Rightarrow |a_n - L| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = L.$$

(b) Suppose the 78 sequence $\{a_n\}$ is bounded monotonically decreasing sequence.

$$M \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \dots \geq m$$

Let L_1 be the g.l.b of $S = \{a_n: n \in \mathbb{N}\}$

Given any $\epsilon > 0$, $L_1 + \epsilon > L_1$ and so $L_1 + \epsilon$ is not a lower bound of $\{a_n\}$

$\Rightarrow \exists$ an integer m , such that

$$a_n \leq a_m < L_1 + \epsilon \quad \forall n \geq m$$

$$\Rightarrow a_n < L_1 + \epsilon \rightarrow (3) \quad \forall n \geq m$$

Also L_1 is g.l.b of $\{a_n\}$

$$a_n \geq L_1 > L_1 - \epsilon \quad \forall n \in \mathbb{N}$$

$$\Rightarrow a_n > L_1 - \epsilon \rightarrow (4) \quad \forall n \geq m$$

By (3) & (4) we have

$$|a_n - L_1| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = L_1$$

$$L_1 \leq \overline{\lim} a_n \quad \forall n$$

$$0 \leq a_n - L_1 \rightarrow (5) \quad \forall n$$

by (3) & (5) $|a_n - L_1| < \epsilon \quad \forall n \geq m$

Remarks # ²⁹ The monotonic convergence theorem establishes the convergence of sequence without knowing the limit in advance. It also gives us a way of calculating the limit by evaluating supremum and infimum. Sometimes the supremum and infimum can not found easily but once we know that it exists, it is often possible to evaluate the limit by other methods.

Sequences defined inductly must be treated differently. If such a sequence is known to converge, then value of the limit can sometimes be determined by inductive relation.

Applications

Q # 1 # prove that the sequence defined

by $a_n = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \dots (1 - \frac{1}{n^2})$
is cgt

Sol clearly $a_n > 0$

$$\begin{aligned} a_{n+1} &= (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2})(1 - \frac{1}{(n+1)^2}) \\ &= a_n \left[1 - \frac{1}{(n+1)^2} \right] < a_n \end{aligned}$$

$\Rightarrow \{a_n\}$ is decreasing and bounded below

It converges to l.u.b which is ≤ 3

$$\therefore \lim_{n \rightarrow \infty} a_n \leq 3$$

$$\therefore a_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$$

$$\geq 2$$

$$\therefore \lim_{n \rightarrow \infty} a_n \geq 2$$

$$\text{Hence } 2 \leq \lim_{n \rightarrow \infty} a_n \leq 3$$

Q#3 prove that the the sequence with general term

$$a_n = 2 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

Converges

Sol # $a_n = 2 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$

$$a_{n+1} = 2 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$= a_n + \frac{1}{(n+1)!} > a_n$$

$\Rightarrow \{a_n\}$ is monotone increasing
using $\frac{1}{n!} < \frac{1}{2^{n-1}}$

$$a_n = 2 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \cancel{\frac{1}{(n+1)!}}$$

$$< 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

Q: 2 # Show that ⁸⁰ the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

Converges. Also $2 \leq \lim a_n \leq 3$

Sol $a_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$$= a_n + \frac{1}{(n+1)!} > a_n \quad \forall n.$$

$\Rightarrow \{a_n\}$ is monotone increasing. We show that $\{a_n\}$ is bounded.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n > 2 \cdot 2 \cdot 2 \cdot \dots \cdot 2 = 2^{n-1}$$

$$\Rightarrow \frac{1}{n!} < \frac{1}{2^{n-1}}$$

using this we have.

$$a_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}$$

$$= 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 + 2 \left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$< 3 \quad \forall n$$

$\Rightarrow \{a_n\}$ increases and is bounded above by 3. So $\{a_n\}$ Converges

$\{a_n\}$ is increasing and ...

$$< 2 + 1 + \frac{1 \left(1 - \left(\frac{1}{2} \right)^n \right)}{1 - \frac{1}{2}}$$

$$= 3 + 2 \left[1 - \left(\frac{1}{2} \right)^n \right] < 4.$$

Hence $\{a_n\}$ is bounded monotone sequence and so converges.

Q #4 prove that the sequence $\{a_n\}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

is dgt

Sol $a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \frac{1}{n+1}$

$$= a_n + \frac{1}{n+1} > a_n$$

$\Rightarrow \{a_n\}$ is increasing sequence.

$$a_1 = 1$$

$$a_2 = 1 + \frac{1}{2}$$

$$a_3 = a_2 + \frac{1}{3} + \frac{1}{4} > \left(\frac{1}{4} + \frac{1}{4} \right) > \frac{1}{2} + a_2$$

$$a_4 = a_3 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{1}{2} + a_3$$

$$a_4 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$q_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \dots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}\right) + \left(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}\right) + \dots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}\right) + \left(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}\right) + \dots + \left(\frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}\right)_{(n-1) \text{ terms}}$$

$$= 1 + \frac{n}{2}$$

$$a_n > 1 + \frac{n}{2}$$

\Rightarrow Sequence a_n is unbounded and hence divergent

Q: 5 # prove that the sequence

$$x_1 = \frac{x_0}{a+x_0}, x_2 = \frac{x_1}{a+x_1}, \dots, x_n = \frac{x_{n-1}}{a+x_{n-1}}$$

, $a > 1, x_0 > 0$ Converges

Sol

$$x_n = \frac{x_{n-1}}{a+x_{n-1}} < x_{n-1}$$

$$\Rightarrow x_n < x_{n-1}$$

\Rightarrow Sequence $\{x_n\}$ is decreasing

2ndly since $a > 1, x_0 > 0$, Therefore all of the terms are +ve which means that sequence is bounded below. Thus the sequence is monotone and bounded. Hence it is cgt.

Q# 6 prove that the sequence with general term

$$a_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \dots + \frac{1}{5^n+1}$$

i.e.

$$a_1 = \frac{1}{5+1} \quad a_2 = \frac{1}{5+1} + \frac{1}{5^2+1}$$

$$a_3 = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} \quad \text{is cgt}$$

Sol

$$a_{n+1} = \frac{1}{5+1} + \frac{1}{5^2+1} + \dots + \frac{1}{5^n+1} + \frac{1}{5^{n+1}+1}$$
$$= a_n + \frac{1}{5^{n+1}+1}$$

$$\Rightarrow a_{n+1} > a_n$$

$\Rightarrow \{a_n\}$ is increasing

Also since $\frac{1}{5^{n+1}} < \frac{1}{5^n} \forall n$.

We have.

$$a_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \dots + \frac{1}{5^n+1}$$

$$< \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots + \frac{1}{5^n}$$

$$= \frac{\frac{1}{5} - \frac{1}{5^{n+1}}}{1 - \frac{1}{5}} = \frac{5}{4} \left[\frac{1}{5} - \frac{1}{5^{n+1}} \right]$$

$$= \frac{1}{4} \left[1 - \frac{1}{5^n} \right] < \frac{1}{4} \forall n.$$

Hence the sequence is convergent.

Q#7* prove that the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

is convergent and its limit lies between $\frac{1}{2}$ and 1

Sol

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$
$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$$
$$> \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1}$$
$$= \frac{2}{2n+2} - \frac{1}{n+1} = 0$$

$$a_{n+1} > a_n$$

$\Rightarrow \{a_n\}$ is monotonically increasing

Also

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$$
$$< \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+1}$$
$$= \frac{n}{n+1} < 1 \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded above by 1

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n}$$
$$> \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

$$a_n > \frac{1}{2}$$

Thus $\frac{1}{2} < \overline{87} a_n < 1 \quad \forall n$
Hence $\frac{1}{2} \leq \lim_{n \rightarrow \infty} a_n \leq 1$

Q: 8 ^{R.G.B}
Ex

Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$
 $\forall n \in \mathbb{N}$. prove that $\{x_n\}$ is increasing
and bounded and hence converges.

Sol we note that

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k} \quad \forall k \geq 2 \quad \text{--- (1)}$$

we will use this fact

$$\begin{aligned} x_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \\ x_{n+1} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \\ &= x_n + \frac{1}{(n+1)^2} > x_n \quad \forall n. \end{aligned}$$

$\Rightarrow \{x_n\}$ is increasing

Also by using (1)

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$$

$$\begin{aligned} &< \frac{1}{1^2} + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \\ &\quad + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \frac{1}{1^2} + \frac{1}{1} - \frac{1}{n} = 2 - \frac{1}{n} < 2 \quad \forall n \end{aligned}$$

49

$\Rightarrow \{x_n\}$ is bounded ⁸⁸
in \mathbb{C}^+

Calculation

Q # (9) # (a) A sequence

$$a_1 = 2, \quad a_{n+1} =$$

prove that $\{a_n\}$ converges

R.G.B. (b) # Let $a > 0$, a_n
which converges to

Let $\delta_1 > 0$ be an

$$s_{n+1} = \frac{1}{2}(s_n +$$

prove that $\{s_n\}$ converges

Note It is general case

Sol # (a) $a_1 = 2$

$$a_2 = \frac{1}{2}$$

$$a_3 = \frac{1}{2}$$

$$a_4 = \frac{1}{2}$$

Casual reader may
decreasing.

OR

$\Rightarrow \{x_n\}$ is bounded and increasing. hence it is cgt

Calculation of square root

Q # (a) # (a) A sequence $\{a_n\}$ is defined by

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$$

R.G.B. prove that $\{a_n\}$ converges and find its limit

(b) # Let $a > 0$, we construct a sequence which converges to \sqrt{a}

Let $\epsilon_1 > 0$ be an arbitrary and.

$$s_{n+1} = \frac{1}{2} \left(s_n + \frac{a}{s_n} \right) \quad \forall n \in \mathbb{N}$$

prove that $\{s_n\}$ converges and find its limit.

Note It is general case of (a)

Sol # (a) $a_1 = 2$.

$$a_2 = \frac{1}{2} \left(2 + \frac{2}{2} \right) = \frac{3}{2} = 1.5$$

$$a_3 = \frac{1}{2} \left(\frac{3}{2} + \frac{2}{\frac{3}{2}} \right) = 1.4167$$

$$a_4 = \frac{1}{2} \left(\frac{17}{12} + \frac{2}{\frac{17}{12}} \right) = \frac{577}{408} \approx 1.4142$$

Casual reader may deduce that $\{a_n\}$ is decreasing.

OR

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$$

$$a_n = \frac{1}{2} \left(a_{n-1} + \frac{2}{a_{n-1}} \right)$$

$$a_n^2 = \frac{1}{4} \left[a_{n-1} + \frac{2}{a_{n-1}} \right]^2$$

$$= \frac{1}{4} \left[\left(a_{n-1} - \frac{2}{a_{n-1}} \right)^2 + 8 \right]$$

$$\geq \frac{1}{4} \left(a_{n-1} - \frac{2}{a_{n-1}} \right)^2 + 2 \geq 2 \quad \forall n \geq 2$$

$$\Rightarrow a_n^2 \geq 2$$

$$\Rightarrow a_n \geq \frac{2}{a_n} \quad \forall n \geq 2$$

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \leq \frac{1}{2} (a_n + a_n) = a_n$$

$$a_{n+1} \leq a_n \quad \forall n \geq 2$$

$\Rightarrow \{a_n\}$ is ultimately decreasing.

OR

$$\text{By } a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$$

$$2a_{n+1}a_n = a_n^2 + 2$$

$$\Rightarrow a_n^2 - (2a_{n+1})a_n + 2 = 0$$

$\Rightarrow a_n$ satisfies the quadratic equation.

$\therefore a_1 = 2 \quad \therefore$ This quadratic equation has

has a real root. 20
 \Rightarrow Disc of it must be non-negative.

$$\Rightarrow 4a_{n+1}^2 - 4(2) \geq 0 \quad \forall n \geq 1$$

$$\Rightarrow a_{n+1}^2 - 2 \geq 0 \quad \forall n \geq 1$$

$$\Rightarrow a_{n+1}^2 \geq 2 \quad \forall n \geq 1$$

$$\Rightarrow a_n^2 \geq 2 \quad \forall n \geq 2.$$

Now

$$a_n - a_{n+1} = a_n - \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$$

$$= a_n - \frac{1}{2} \left(\frac{a_n^2 + 2}{a_n} \right)$$

$$= \frac{2a_n^2 - a_n^2 - 2}{2a_n} \quad \forall n \geq 2$$

$$\geq \frac{1}{2} \left(\frac{a_n^2 - 2}{a_n} \right) \geq 0 \quad \because a_n^2 \geq 2, a_n > 0 \quad \forall n$$

$$\Rightarrow a_{n+1} \leq a_n \quad \forall n \geq 2$$

$\Rightarrow \{a_n\}$ is decreasing and is bounded below by 0

$\Rightarrow \lim_{n \rightarrow \infty} a_n$ exists

let $\lim_{n \rightarrow \infty} a_n = l$

..... $\lim_{n \rightarrow \infty} a_n$ is a the root of $x^2 - 2x - 1 = 0$

$$\begin{aligned} \therefore a_n &> 0 \quad \forall n \\ \therefore l &\geq 0 \end{aligned}$$

$$\lim_{n \rightarrow \infty} a_{n+1} = \frac{1}{2} \left(\lim_{n \rightarrow \infty} a_n + \frac{2}{\lim_{n \rightarrow \infty} a_n} \right)$$

$$l = \frac{1}{2} \left(l + \frac{2}{l} \right) = \frac{l}{2} + \frac{1}{l}$$

$$2l^2 = l^2 + 2$$

$$l^2 = 2 \Rightarrow l = \pm\sqrt{2}$$

$$\therefore l \geq 0$$

\therefore only possibility is $l = \sqrt{2} \approx 1.414213$

$$(b) \quad b_{n+1} = \frac{1}{2} \left(b_n + \frac{9}{b_n} \right)$$

$$b_n = \frac{1}{2} \left(b_{n-1} + \frac{9}{b_{n-1}} \right)$$

$$\Rightarrow b_n^2 = \frac{1}{4} \left(b_{n-1} + \frac{9}{b_{n-1}} \right)^2$$

$$= \frac{1}{4} \left[\left(b_{n-1} - \frac{9}{b_{n-1}} \right)^2 + 4 \cdot 9 \right]$$

$$= \frac{1}{4} \left(b_{n-1} - \frac{9}{b_{n-1}} \right)^2 + 9 \geq 9 \quad \forall n \geq 2$$

$$\Rightarrow b_n^2 \geq 9$$

$$\Rightarrow b_n \geq \frac{9}{b_n} \Rightarrow \frac{9}{b_n} \leq b_n \quad \forall n \geq 2$$

$$\therefore s_n > 0 \quad \forall n$$

$$l > 0$$

$$s_{n+1} = \frac{1}{2} \left(s_n + \frac{9}{s_n} \right)$$

$$\lim_{n \rightarrow \infty} s_{n+1} = \frac{1}{2} \left(\lim_{n \rightarrow \infty} s_n + \frac{9}{\lim_{n \rightarrow \infty} s_n} \right)$$

$$l = \frac{1}{2} \left(l + \frac{9}{l} \right)$$

$$2l^2 = l^2 + 9$$

$$l^2 = 9$$

$$l = \sqrt{9}$$

Note For the purpose of calculation, it is often important to have an estimate of how rapidly the sequence s_n converges to $\sqrt{9}$.

$$\therefore \lim_{n \rightarrow \infty} s_n = \inf_n s_n = \sqrt{9}$$

$$\therefore \sqrt{9} \leq s_n \quad \forall n \geq 2$$

$$\Rightarrow 9 \leq s_n^2 \quad \forall n \geq 2$$

$$\Rightarrow \frac{9}{s_n} \leq s_n \quad \forall n \geq 2 \quad \text{or} \quad \frac{9}{s_n} \leq \sqrt{9} \leq s_n$$

$$0 \leq s_n - \sqrt{9} \leq s_n - \frac{9}{s_n} \quad \because \frac{9}{s_n} \leq \sqrt{9}$$

$$0 \leq \frac{s_n^2 - 9}{s_n} \leq \frac{s_n^2 - 9}{s_n} \quad \forall n \geq 2$$

Using this inequality we can calculate $\sqrt{9}$ to any desired degree of accuracy.

Thus $\lim_{n \rightarrow \infty} x_n$ is a root of $x^2 - 2x - 1 = 0$

$$s_{n+1} = \frac{1}{2} \left(s_n + \frac{9}{s_n} \right) \leq \frac{1}{2} (s_n + s_n) \quad \forall n \geq 2$$

$$s_{n+1} \leq s_n \quad \forall n \geq 2$$

$\Rightarrow \{s_n\}$ is decreasing

OR

$$s_{n+1} = \frac{1}{2} \left(s_n + \frac{9}{s_n} \right)$$

$$\Rightarrow s_n^2 - (2s_{n+1})s_n + 9 = 0$$

$\Rightarrow s_n$ satisfies quadratic equation & $s_1 > 0$

\Rightarrow This equation has real roots.

$$\Rightarrow \text{Disc} \geq 0$$

$$4s_{n+1}^2 - 49 \geq 0$$

$$s_{n+1}^2 \geq \frac{49}{4} \quad \forall n \geq 1$$

$$\text{or } s_n^2 \geq \frac{49}{4} \quad \forall n \geq 2$$

$$s_n - s_{n+1} = \frac{1}{2} \left(\frac{s_n^2 - a}{s_n} \right) \geq 0 \quad \forall n \geq 2$$

$$\Rightarrow s_{n+1} \leq s_n \quad \forall n \geq 2$$

$$\because s_n^2 \geq \frac{49}{4}$$

$$s_n > 0 \quad \forall n$$

$\Rightarrow \{s_n\}$ is ultimately decreasing.

Also $s_n \geq 0 \quad \forall n$.

Thus $\{s_n\}$ monotone bounded and.

hence is cgt. Let $\lim_{n \rightarrow \infty} s_n = l$.

Exercise Show ⁹⁴ that the sequence defined by $a_{n+1} = \frac{1}{2}(a_n + \frac{9}{a_n})$ $n \geq 1, a_1 > 0$ Converges to 3

Q # 10 ^{G.R.B (exaple)} # prove that the sequence defined by $x_1 = 2, x_{n+1} = 2 + \frac{1}{x_n} \quad \forall n \in \mathbb{N}$ is convergent and converges to a +ve root of equation $x^2 - 2x - 1 = 0$

Sol # $x_{n+1} = 2 + \frac{1}{x_n} > 2 \quad \forall n \geq 1$
 $\& x_1 = 2$
 $\Rightarrow x_n \geq 2 \quad \forall n \geq 1$

Also $x_1 = 2$
 $x_2 = 2 + \frac{1}{x_1} = 2 + \frac{1}{2} = \frac{5}{2} = 2.5$
 $x_3 = 2 + \frac{1}{x_2} = 2 + \frac{2}{5} = \frac{12}{5} = 2.4$
 $x_4 = 2 + \frac{1}{x_3} = 2 + \frac{5}{12} = \frac{29}{12}$
 $x_5 = 2 + \frac{12}{29} = \frac{70}{29}$

We note that sequence ultimately decrease.
 $\Rightarrow \{x_n\}$ bounded monotone and hence Cgt.

Let $\lim_{n \rightarrow \infty} x_n = l$. Then
 $l = 2 + \frac{1}{l} \Rightarrow l^2 - 2l - 1 = 0$
 Thus $\lim_{n \rightarrow \infty} x_n$ is a +ve root of $x^2 - 2x - 1 = 0$

and $\lim_{n \rightarrow \infty} x_n = \frac{95}{1 + \sqrt{2}}$

Q# 11 (R.G.B) Let $\{y_n\}$ be defined by
 $y_1 = 1$ $y_{n+1} = \frac{1}{4}(2y_n + 3)$ for $n \geq 1$

Show that $\lim_{n \rightarrow \infty} y_n = \frac{3}{2}$

Sol By direct calculation.

$$y_2 = \frac{5}{4}$$

Hence

$$y_1 < y_2 < 2$$

We show by induction that $y_n < 2 \quad \forall n \in \mathbb{N}$

It is true for $n=1, 2$

Let $y_k < 2$ for some $k \in \mathbb{N}$, $k \geq 2$

$$\text{Then } y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2 \cdot 2 + 3) = \frac{7}{4} < 2$$

$$\Rightarrow y_{k+1} < 2 \Rightarrow y_n < 2 \quad \forall n \in \mathbb{N}$$

We show by induction that $y_n < y_{n+1}$

$$\text{For } n=1 \quad y_2 = \frac{5}{4} = 1.25 > 1 = y_1$$

$$\Rightarrow y_1 < y_2$$

Let $y_k < y_{k+1}$ for some $k \in \mathbb{N}$

$$\text{Then } 2y_k < 2y_{k+1}$$

$$2y_k + 3 < 2y_{k+1} + 3$$

$$y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2y_{k+1} + 3) = y_{k+2}$$

Thus $g_n < 2 \quad \forall n \geq 2$

Also

$$g_1 = 1$$

\Rightarrow

$$1 \leq g_n < 2 \quad \forall n \in \mathbb{N}$$

$\Rightarrow \{g_n\}$ is bounded.

OR

$$g_2 = \sqrt{2} < 2$$

$$g_3 = \sqrt{2g_2} < \sqrt{2 \cdot 2} = 2$$

$$g_4 = \sqrt{2g_3} < \sqrt{2 \cdot 2} = 2$$

$$g_5 = \sqrt{2g_4} < \sqrt{2 \cdot 2} = 2$$

$$\Rightarrow g_n < 2 \quad \forall n \geq 2.$$

OR

$$g_1 = 1$$

$$g_2 = \sqrt{2g_1} = \sqrt{2} = 2^{\frac{1}{2}}$$

$$g_3 = \sqrt{2g_2} = \sqrt{2 \cdot \sqrt{2}} = 2^{\frac{1}{2} + \frac{1}{4}} = 2^{\frac{1}{2} + \frac{1}{2^2}}$$

$$g_4 = \sqrt{2g_3} = \left(2 \cdot 2^{\frac{1}{2} + \frac{1}{2^2}}\right)^{\frac{1}{2}} = 2^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}}$$

$$g_n = 2^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}}$$

$$g_{n+1} = 2^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} + \frac{1}{2^n}} < 2$$

$$= g_n \cdot 2^{\frac{1}{2^n}} \geq g_n$$

Also

Thus $y_k < y_{k+1} \xrightarrow{96}$
 $\Rightarrow y_n < y_{n+1} \Rightarrow y_{k+1} < y_{k+2}$
 $\Rightarrow \{y_n\}$ is increasing and bounded above by 2
 $\Rightarrow \{y_n\}$ is cgt.

$$\text{Let } \lim_{n \rightarrow \infty} y_n = l = \lim_{n \rightarrow \infty} y_{n+1}$$

$$\text{Then from } y_{n+1} = \frac{1}{4}(2y_n + 3)$$

$$l = \frac{1}{4}(2l + 3)$$

$$4l = 2l + 3$$

$$2l = 3 \Rightarrow l = \frac{3}{2} = 1.5$$

Q # 12 (R.G.B) Show that the sequence

of real numbers defined by

$$z_1 = 1 \quad z_{n+1} = \sqrt{2z_n} \text{ Converges}$$

$$\text{and } \lim_{n \rightarrow \infty} z_n = 2.$$

Sol $z_1 = 1 \quad z_2 = \sqrt{2} < 2$

$$\Rightarrow z_1 < z_2 < 2 \rightarrow \textcircled{1}$$

for $n = 2$, we have

$$z_2 = \sqrt{2} < 2$$

$$\text{Let } z_k < 2 \quad \text{for } k \geq 2, k \in \mathbb{N}$$

$$\text{Then } z_{k+1} = \sqrt{2z_k} < \sqrt{2 \cdot 2} = 2$$

Q: 12 Show that the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2a_n}$ Converges to 2

Sol

$$a_1 = \sqrt{2} < 2.$$

$$a_2 = \sqrt{2a_1} < \sqrt{2 \cdot 2} < 2.$$

$$a_3 = \sqrt{2a_2} < \sqrt{2 \cdot 2} < 2.$$

$$a_n < 2 \quad \forall n.$$

Also $a_1 = \sqrt{2}$

$$a_2 = \sqrt{2\sqrt{2}} > \sqrt{2} = a_1.$$

$$a_2 > a_1.$$

Let $a_{k+1} > a_k \rightarrow$ ① for some $k \in \mathbb{N}$

Then $a_{k+2} = \sqrt{2a_{k+1}}$ & $a_{k+1} = \sqrt{2a_k}$

From ①

$$2a_{k+1} > 2a_k$$

$$\Rightarrow \sqrt{2a_{k+1}} > \sqrt{2a_k}$$

$$a_{k+2} > a_{k+1}$$

$$a_{k+1} > a_k \Rightarrow a_{k+2} > a_{k+1}$$

$\Rightarrow \{z_n\}$ is bounded ¹⁸ and increasing.

OR
Increasing fact can be proved by induction as

$$z_1 < z_2 \rightarrow \textcircled{1}$$

We are to prove that

$$z_n < z_{n+1} \quad \forall n.$$

For $n=1$ inequality is true from $\textcircled{1}$

Let it be true for $n=k$ i.e.

$$z_k < z_{k+1} \rightarrow \textcircled{2}$$

Now $z_{k+1} = \sqrt{2z_k} < \sqrt{2z_{k+1}} = z_{k+2}$ by $\textcircled{2}$

$$\Rightarrow z_{k+1} < z_{k+2}$$

$$\text{Thus } z_k < z_{k+1} \Rightarrow z_{k+1} < z_{k+2}.$$

$$\Rightarrow z_n < z_{n+1} \quad \forall n.$$

$\Rightarrow \{z_n\}$ increasing sequence & bounded.

So $\{z_n\}$ is cgt. Let $\lim_{n \rightarrow \infty} z_n = l$.

$$\text{Then } z_{n+1} = \sqrt{2z_n}$$

$$\Rightarrow l = \sqrt{2l}.$$

$$l^2 = 2l.$$

$$\Rightarrow l^2 - 2l = 0$$

$$\Rightarrow l = 0$$

$$l = 2.$$

$$l(l-2) = 0$$

$$\text{But } 1 \leq l \leq 2$$

$$\Rightarrow \boxed{l = 2}$$

150

2

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2 \sqrt{2}} = 2 \cdot 2^{\frac{1}{4}} = 2^{\frac{1}{2} + \frac{1}{2^2}}$$

$$a_3 = \sqrt{2 \sqrt{2 \sqrt{2}}} = (2 \cdot 2^{\frac{1}{2} + \frac{1}{4}})^{\frac{1}{2}} = 2^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}}$$

$$= 2$$

$$= 2$$

$$\frac{1}{2^2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$$

$$a_n =$$

$$\frac{1}{2^2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$a_{n+1} =$$

$$a_{n+1} =$$

$$2^{\frac{1}{2^{n+1}}} \geq a_n \quad \forall n$$

$\Rightarrow \{a_n\}$ is increasing

Again

$$\frac{1}{2} + \dots + \frac{1}{2^n}$$

$$a_n < 2$$

$$< 2 \quad \forall n$$

$$\Rightarrow a_n < 2$$

$\Rightarrow \{a_n\}$ is bounded monotone and hence convergent. Let $\lim_{n \rightarrow \infty} a_n = l$

$$\begin{aligned} &\because 2^{\frac{1}{n}} > 1 \quad \forall n > 1 \\ &\therefore 2^{\frac{1}{2^{n+1}}} > 1 \\ &\Rightarrow a_n \cdot 2^{\frac{1}{2^{n+1}}} > a_n \\ &a_{n+1} > a_n \end{aligned}$$

100
OR

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} = 2^{\frac{1}{2} + \frac{1}{4}}$$

$$a_3 = \sqrt{2a_2} = \left(2 \cdot 2^{\frac{1}{2} + \frac{1}{4}}\right)^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}} = 2^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}$$

$$a_n = 2^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}}$$

$$a_{n+1} = 2^{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}}}$$

$$a_{n+1} = a_n \cdot 2^{\frac{1}{2^{n+1}}} \geq a_n \quad \forall n$$

$\Rightarrow \{a_n\}$ is increasing

Again,

$$a_n = 2^{\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}} < 2 \quad \forall n$$

$$\Rightarrow a_n < 2 \quad \forall n$$

$\Rightarrow \{a_n\}$ bounded monotone and hence is convergent. Let $\lim_{n \rightarrow \infty} a_n = l$

$$\begin{aligned} &\because 2^{\frac{1}{n}} > 1 \quad \forall n > 1 \\ &\therefore 2^{\frac{1}{2^{n+1}}} > 1 \\ &\Rightarrow a_n \cdot 2^{\frac{1}{2^{n+1}}} > a_n \\ &a_{n+1} > a_n \end{aligned}$$

$$\lim_{n \rightarrow \infty} a_n = l$$

$$\Rightarrow a_n > l$$

$$\Rightarrow (\lim_{n \rightarrow \infty} a_n = l)$$

$$l = 0$$

$$a_n > l$$

$$l = 2$$

Show that
by
to 3

$$s_2 =$$

$$s_3 =$$

$$s_4 =$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

$$= \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

$$= a_n$$

$$\Rightarrow [k]$$

$$\forall n$$

$$\lim_{n \rightarrow \infty} a_n$$

$$\text{sequence}$$

$$= \sqrt{3} \cdot \frac{1}{2^n}$$

$$= \frac{1}{2^n}$$

$$= \sqrt{3 \cdot 3}$$

$$\frac{1}{2^2} + \frac{1}{2^3}$$

$$\frac{1}{2^3} + \dots$$

$$+ \dots + \frac{1}{2^n}$$

$$> a_n$$

Then $\lim_{n \rightarrow \infty} a_{n+1} = l$

$$a_{n+1} = \sqrt{2a_n}$$

$$\Rightarrow a_{n+1}^2 = 2a_n$$

$$\Rightarrow \left(\lim_{n \rightarrow \infty} a_{n+1} \right)^2 = 2 \lim_{n \rightarrow \infty} a_n$$

$$l^2 = 2l \Rightarrow l(l-2) = 0$$

$$\Rightarrow l = 0 \text{ or } l = 2$$

But $a_n > 0 \quad \forall n$.

$$\Rightarrow l = 2 \Rightarrow \lim_{n \rightarrow \infty} a_n = 2.$$

Q: 13 Show that the sequence $\{s_n\}$ defined by $s_{n+1} = \sqrt{3s_n}$ $s_1 = 1$

Converges to 3

Sol

$$s_1 = 1$$

$$s_2 = \sqrt{3s_1} = \sqrt{3} = 3^{\frac{1}{2}}$$

$$s_3 = \sqrt{3s_2} = \sqrt{3 \cdot 3^{\frac{1}{2}}} = 3^{\frac{1}{2} + \frac{1}{2^2}}$$

$$s_4 = \frac{1}{3} + \frac{1}{2^2} + \frac{1}{2^3}$$

$$s_n = \frac{1}{3} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}}$$

$$\Rightarrow s_{n+1} = \frac{1}{3} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}} + \frac{1}{2^n}$$

$$= a_n \cdot 3^{\frac{1}{2^n}} > a_n \quad \forall n.$$

$$\Rightarrow \boxed{l = 2}$$

Sol #

49

1.3

$$x_n =$$

$n-1$

$$x_1 = x_1$$

$$x_1 = \sqrt{2+1}$$

Suppose $x_k > x_{k-1}$ for $k \in \mathbb{N}$

$$\Rightarrow x_k > 2 +$$

$$\Rightarrow x_k > \sqrt{2+}$$

$$\Rightarrow x_{k+1} > x_{k+1-1}$$

Thus $x_1 > x_n$ \forall induction.

Now $= 1 < 2$

$$x_1 + x_1 < \sqrt{2+} \quad 2$$

$$x_1 + x_2 < \sqrt{2+} \quad 2$$

$$x_1 + x_3 < \sqrt{2+} \quad 2$$

OR $x_1 <$

OR

By $x_k < 2$

Let $x_k < 4$

Then $x_k < 2$

$\Rightarrow x_k < 2$

$\Rightarrow x_1 < 2$

$$x_1 + 1 < \sqrt{2+x_k}$$

$$< \sqrt{2+2}$$

hence $x_n < 2 \quad \forall n$

$$1/17 + 9$$

Sol #

$$x_1 = 1$$

1.3

$$x_n = \sqrt{2 + x_{n-1}}$$

$$x_2 = \sqrt{2 + x_1} = \sqrt{2 + 1} = \sqrt{3} > 1 = x_1$$

$$\therefore x_2 > x_1$$

Suppose that $x_k > x_{k-1}$ for $k \in \mathbb{N}$

$$\Rightarrow 2 + x_k > 2 + x_{k-1}$$

$$\Rightarrow \sqrt{2 + x_k} > \sqrt{2 + x_{k-1}}$$

$$\Rightarrow x_{k+1} > x_k = x_{k+1-1}$$

Thus $x_{n+1} > x_n \quad \forall n$ by induction.

Now

$$x_1 = 1 < 2$$

$$x_2 = \sqrt{2 + x_1} < \sqrt{2 + 2} < 2$$

$$x_3 = \sqrt{2 + x_2} < \sqrt{2 + 2} < 2$$

$$x_4 = \sqrt{2 + x_3} < \sqrt{2 + 2} < 2$$

$$\dots \dots \dots x_n < 2 \quad \forall n.$$

OR

$$x_1 < 2$$

OR

By induction

$$x_k < 2$$

$$x_{k+1} = \sqrt{2 + x_k}$$

Let

$$2 + x_k < 4$$

$$< \sqrt{2 + 2} < 2$$

Then

$$\sqrt{2 + x_k} < 2$$

Hence

$$x_n < 2 \quad \forall n$$

\Rightarrow

$$x_{k+1} < 2$$

\Rightarrow

$$\dots \dots \dots x_n < 2 \quad \forall n$$

Also $\frac{102}{2}$

$$S_{n+1} = 3 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} \right) < 3$$

$$\therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}} < 1$$

$$\Rightarrow S_{n+1} < 3 \quad \forall n$$

Hence $\{S_n\}$ is bounded and converges.

$$\text{Let } \lim_{n \rightarrow \infty} S_n = l$$

$$S_n \geq 1 \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} S_n \geq 1$$

$$S_{n+1} = \sqrt{3 S_n}$$

$$S_{n+1}^2 = 3 S_n$$

$$\Rightarrow \left(\lim_{n \rightarrow \infty} S_{n+1} \right)^2 = 3 \lim_{n \rightarrow \infty} S_n$$

$$l^2 = 3l \Rightarrow l(l-3) = 0$$

$$\Rightarrow l = 0 \text{ or } l = 3$$

$$\therefore \lim_{n \rightarrow \infty} S_n \geq 1 \Rightarrow \lim_{n \rightarrow \infty} S_n = 3$$

Q# 14 If $S_{n+1} = \sqrt{7 S_n}$ $S_1 = 1$, prove that $\{S_n\}$ is convergent. What is its limit.
try yourself.

Q: 15 ^{R.G.B} Show that the sequence $\{x_n\}$, where $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}$ $\forall n \geq 2$ is convergent and converges to 2.

Thus $\{x_n\}$ bounded ¹⁰⁴ monotone sequence.
and $1 \leq a_n < 2$.

$$\Rightarrow 1 \leq \lim_{n \rightarrow \infty} a_n \leq 2$$

Let $\lim_{n \rightarrow \infty} x_n = l$.

$$x_n \geq \sqrt{2+x_{n-1}} \Rightarrow x_n^2 \geq 2+x_{n-1}$$

$$\Rightarrow (\lim x_n)^2 \geq 2 + \lim x_{n-1}$$

$$l^2 = 2 + l$$

$$\Rightarrow l^2 - l - 2 = 0$$

$$\Rightarrow (l-2)(l+1) = 0$$

$$\Rightarrow l = 2 \quad l = -1$$

$$\because x_n > 0 \quad \forall n \Rightarrow l = 2$$

Q.16 prove that the sequence $\{a_n\}$

defined by $a_1 = \sqrt{7}$ & $a_{n+1} = \sqrt{7+a_n}$
Converges to the true square root of
 $x^2 - x - 7 = 0$

Sol $a_1 = \sqrt{7}$ $a_{n+1} = \sqrt{7+a_n}$

$$a_2 = \sqrt{7+a_1} = \sqrt{7+\sqrt{7}} > \sqrt{7} = a_1$$

Let $a_k > a_{k-1}$ for some $k \in \mathbb{N}$

$$\begin{aligned} 7+a_k &> 7+a_{k-1} \\ \Rightarrow \sqrt{7+a_k} &> \sqrt{7+a_{k-1}} \Rightarrow a_{k+1} > a_k \end{aligned}$$

105

By Mathematical induction.

$\Rightarrow \{a_n\}$ is monotonically increasing $\forall n$.

Now $a_1 = \sqrt{7} < 7$
Let $a_k < 7$ for some $k \in \mathbb{N}$

$$7 + a_k < 7 + 7 = 14$$

$$\sqrt{7 + a_k} < \sqrt{14} < \sqrt{49} = 7$$

$$\Rightarrow a_{k+1} < 7$$

\Rightarrow By mathematical induction.

$$a_n < 7 \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded above

Since $\{a_n\}$ is monotonically increasing and bounded above, it is cgt.

Let $\lim_{n \rightarrow \infty} a_n = l$

Now $a_{n+1} = \sqrt{7 + a_n} \Rightarrow a_{n+1}^2 = 7 + a_n$

$$\Rightarrow l^2 = 7 + l \Rightarrow l^2 - l - 7 = 0$$

$$\therefore a_n > 0 \quad \forall n$$

$\therefore \{a_n\}$ converges to a real root of $x^2 - x - 7 = 0$

$$l = \frac{1 \pm \sqrt{1+28}}{2} = \frac{1 \pm \sqrt{29}}{2}$$

But $\frac{1 - \sqrt{29}}{2} < 0$ so $l = \frac{1 + \sqrt{29}}{2}$

By Mathematical induction

$\Rightarrow \{a_n\}$ is monotonically increasing $\forall n$

Now

Let

$$a_1 = \sqrt{7} < 7$$

$$a_k < 7$$

for some $k \in \mathbb{N}$

$$7 + a_k < 7 + 7 = 14$$

$$\sqrt{7 + a_k} < \sqrt{14} < \sqrt{49} = 7$$

$$\Rightarrow a_{k+1} < 7$$

\Rightarrow By mathematical induction.

$$a_n < 7 \quad \forall n$$

$\{a_n\}$ is bounded above

Since $\{a_n\}$ is monotonically increasing and bounded above, it is cgt.

Let $\lim_{n \rightarrow \infty} a_n = l$

Now $a_{n+1} = \sqrt{7 + a_n} \Rightarrow a_{n+1}^2$

$$l^2 = 7 + l \Rightarrow l^2 - l - 7 = 0$$

$$a_n > 0 \quad \forall n$$

$\{a_n\}$ Converges to a true root of

$$= \frac{1 \pm \sqrt{1+28}}{2} = \frac{1 \pm \sqrt{29}}{2}$$

$$\frac{1 - \sqrt{29}}{2} < 0 \quad \text{So } l = \frac{1 + \sqrt{29}}{2}$$

Thus $\{a_n\}$ as $a_1 =$ root of

Thus x_1

$$x_1^2$$

$$x_1^2$$

$$x_1^2$$

$$\sqrt{x_1^2}$$

$\Rightarrow \{x_n\}$ is mon

Now

$$x_n^2$$

$$x_n^2$$

$$x_n^2$$

$$\Rightarrow x_n >$$

$$x_n$$

$$x_1 > 0$$

$$\text{Let } x_k >$$

$$\sqrt{a + x_k} >$$

$$a_{k+1} >$$

Thus $x_n >$

$$x_1 < \alpha$$

Thus

$$\frac{107}{x_1 > \alpha}$$

$$x_1 > -\beta$$

$$\Rightarrow x_1 > \alpha$$

Thus $\{x_n\}$ is increasing or decreasing according as $x_1 = \sqrt{a}$ is less or greater than the eve. root of the equation.

$$x^2 - x - a = 0 \text{ or } x = \frac{1 \pm \sqrt{1+4a}}{2}$$

Thus $x_1 > \alpha \Rightarrow \textcircled{2}$ i.e.

$$x_1^2 - x_1 - a > 0$$

$$x_1^2 > x_1 + a$$

$$x_1 > \sqrt{x_1 + a}$$

$$\sqrt{x_1 + a} < x_1 \Rightarrow x_2 < x_1$$

$\Rightarrow \{x_n\}$ is monotonically decreasing

$$\text{Now } x_n^2 = x_{n-1} + a > x_n + a \quad \because x_n < x_{n-1}$$

$$x_n^2 - x_n - a > 0$$

$$\text{from } \textcircled{2} \Rightarrow x_n > \alpha \quad \forall n$$

OR

$$x_1 > \alpha$$

for some $k \in \mathbb{N}$

$$\text{Set } x_k > \alpha$$

$$\sqrt{a + x_k} > \sqrt{a + \alpha}$$

$$\because x_k > \alpha$$

$$x_1 > \alpha$$

$$x_1^2 > \alpha^2$$

$$a > \alpha^2$$

$$x_{k+1} > \sqrt{a + \alpha} > \alpha$$

Thus $x_n > \alpha \quad \forall n$ by induction.

Q #17 (R.G.B) LOG

If $x_1 = \sqrt{a}$ ($a > 0$) & $x_{n+1} = \sqrt{a+x_n}$
 $\forall n \in \mathbb{N}$ show that $\{x_n\}$ converges and
 find its limit.

Sol

$$x_{n+1}^2 - x_n^2 = (a+x_n) - (a+x_{n-1})$$

$$= x_n - x_{n-1}$$

Thus we note that

$$x_n > x_{n-1} \Rightarrow x_{n+1} > x_n$$

$$\text{and } x_n < x_{n-1} \Rightarrow x_{n+1} < x_n.$$

Thus $\{x_n\}$ is a monotone sequence.
 Also $\{x_n\}$ is an increasing or decreasing according

as

$$x_2 > x_1$$

$$\sqrt{a+x_1} > x_1$$

$$\Rightarrow a+x_1 > x_1^2$$

$$x_1^2 - x_1 - a < 0 \rightarrow (1)$$

Now product of roots of $x_1^2 - x_1 - a = 0$

$$= -\frac{a}{1} < 0$$

\Rightarrow one of roots is -ve and other is +ve.

$$x_1 = \frac{1 \pm \sqrt{4a+1}}{2}$$

(1)

$$(x_1 - \alpha)(x_1 + \beta) < 0$$

$$\Rightarrow x_1 - \alpha < 0$$

$$\therefore x_1 > 0$$

$$x_1 + \beta > 0$$

$$x_1 < \alpha = \frac{1 + \sqrt{4a+1}}{2}$$

or

$$x_2 < x_1$$

$$\sqrt{a+x_1} < x_1$$

$$a+x_1 < x_1^2$$

$$x_1^2 - x_1 - a > 0 \rightarrow (2)$$

$$\alpha = \frac{1 + \sqrt{4a+1}}{2} > 0$$

$$-\beta = \frac{1 - \sqrt{4a+1}}{2} < 0 \quad \beta > 0$$

$$(2) \Rightarrow (x_1 - \alpha)(x_1 + \beta) > 0$$

$$\Rightarrow x_1 - \alpha > 0 \text{ \& } x_1 + \beta > 0$$

49
MAR
Thus $\{x_n\}$ is ¹⁰⁸ monotonically decreasing sequence which is bounded below.

Hence $\lim_{n \rightarrow \infty} x_n$ exists. Let $\lim_{n \rightarrow \infty} x_n = l$.
We have $l \geq \alpha$.

Now

$$x_{n+1} = \sqrt{a + x_n}$$

$$\Rightarrow l^2 = l + a$$

$$l^2 - l - a = 0$$

$$l = \frac{1 \pm \sqrt{1+4a}}{2} \Rightarrow l = \frac{1 + \sqrt{4a+1}}{2} = \alpha > 0$$

Similarly

$$x_1 < \alpha$$

$$\Rightarrow x_1^2 - x_1 - a < 0 \quad \text{from ①}$$

$$x_1 < \sqrt{x_1 + a} = x_2$$

$$x_1 < x_2$$

$\Rightarrow \{x_n\}$ is monotonically increasing

$$\text{Now } x_n^2 = x_{n-1} + a < x_n + a$$

$$x_n^2 - x_n - a < 0$$

$$\Rightarrow x_n < \alpha \quad \forall n \quad \text{by ①}$$

Thus $\{x_n\}$ is monotonically ^{increasing} ~~decreasing~~ sequence & bounded above by α .

Hence its limit exist & $\lim_{n \rightarrow \infty} x_n = \frac{1 + \sqrt{4a+1}}{2} = \alpha$.

By Muhammad Hussain
Assistant Professor Asghar Mall College.

Q. 18 (R.G.B) 109¹⁰⁶

Let $a > 0$ & let $a_1 > 0$. Define.

$$a_{n+1} = \sqrt{a + a_n} \quad \forall n \in \mathbb{N}$$

Show that $\{a_n\}$ is cgt and find the limit

Sol #
$$a_{n+1}^2 - a_n^2 = (a + a_{n+1}) - (a + a_n)$$

$$= a_{n+1} - a_n$$

$$\Rightarrow a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n}$$

Since $a_{n+1} + a_n > 0$, it follows that $a_{n+1} - a_n$

and $a_n - a_{n-1}$ have same sign

$$\text{i.e. } a_{n+1} \geq a_n \text{ iff } a_n \geq a_{n-1}$$

$$\Rightarrow \{a_n\} \text{ increasing or decreasing according to } a_2 > a_1 \text{ or } a_2 < a_1.$$

Case I If $a_2 > a_1$, then $\{a_n\}$ increasing by Mathematical induction.

Also

$$\begin{aligned} a_{n+1}^2 - a_n^2 &= a_{n+1} - a_n \\ &= (a_n^2 - a_n + a) \rightarrow (1) \end{aligned}$$

$$a_{n+1} > a_n \quad \forall n$$

$$\sqrt{a + a_n} > a_n \quad \forall n$$

$$\Rightarrow a + a_n > \overset{110}{a_n^2}$$

$$0 > a_n^2 - a_n - a$$

$$\text{or } a_n^2 - a_n - a < 0 \rightarrow (2)$$

$$\therefore a_n > 0 \quad \forall n \geq 1$$

$\therefore a_n$ is the true root of equation.

$$a_n^2 - a_n - a = 0$$

Let true root be α , then $\alpha = \frac{1 + \sqrt{4a+1}}{2}$

$$\text{Then other root} = \frac{\text{product of roots}}{\alpha}$$

$$= -\frac{a}{\alpha}$$

$$a_n^2 - a_n - a = (a_n - \alpha) \left(a_n + \frac{a}{\alpha}\right)$$

from (2)

$$a_n^2 - a_n - a < 0$$

$$\Rightarrow (a_n - \alpha) \left(a_n + \frac{a}{\alpha}\right) < 0$$

$$\therefore a_n + \frac{a}{\alpha} > 0$$

$$\therefore a_n - \alpha < 0$$

$$\Rightarrow a_n < \alpha \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded above by the root of

$$x^2 - x - a = 0$$

Also $0 < a_1 < \alpha$. Then $\{a_n\}$ is increasing

Thus $\{a_n\}$ is increasing when a_1 is less than

the root of $x^2 - x - a = 0$ i.e. less than $\alpha = \frac{1 + \sqrt{4a+1}}{2}$

Thus when $a_1 \leq \alpha$ III

Then $\{a_n\}$ bounded and increasing
& hence is cgt

Case II If $a_2 < a_1$, then $\{a_n\}$ is decreasing
by Mathematical induction. i.e.

$$a_{n+1} < a_n$$

$$\Rightarrow \frac{1}{a+a_n} < a_n$$

$$\Rightarrow a+a_n < a_n^2$$

$$\Rightarrow a_n^2 - a_n - a > 0$$

$$\Rightarrow (a_n - \alpha) \left(a_n + \frac{a}{\alpha}\right) > 0$$

$$\therefore a_n + \frac{a}{\alpha} > 0$$

$$\therefore a_n - \alpha > 0$$

$$\Rightarrow a_n > \alpha \quad \forall n.$$

$$\Rightarrow \{a_n\} \text{ is bounded below by } \alpha = \frac{1 + \sqrt{4a+1}}{2}$$

Also $a_1 \geq \alpha$

Thus if $a_1 > \alpha$, then $\{a_n\}$ is decreasing
and bounded and hence cgt.

Let $\lim_{n \rightarrow \infty} a_n = l$.

$$\Rightarrow \lim_{n \rightarrow \infty} a_{n+1} = \overline{\lim_{n \rightarrow \infty} a_n + a}$$
$$l = \overline{l + a}.$$

$$\Rightarrow l^2 = l + a$$

$$\Rightarrow l^2 - l - a = 0$$

$$\Rightarrow l = \frac{1 \pm \sqrt{1+4a}}{2}$$

$$\because a_n > 0 \quad \forall n \quad \therefore \lim_{n \rightarrow \infty} a_n \geq 0$$

$$\Rightarrow l = \frac{1 + \sqrt{1+4a}}{2} = \alpha \text{ the root of equation } x^2 - x - a = 0$$

Q: 19 (Grashil exercise)

Define a sequence by

$$a_1 = k, (k > 0), \quad a_{n+1} = \sqrt{k + a_n} \quad n \geq 1$$

Show that $\{a_n\}$ has a limit and find it.

Sol # $a_n > 0 \quad \forall n$.

$$a_2 = \sqrt{a_1 + k} = \sqrt{k + k} = \sqrt{2k}$$

$$\begin{aligned} a_{n+1}^2 - a_n^2 &= (a_n + k) - (a_{n-1} + k) \\ &= a_n - a_{n-1} \end{aligned}$$

$$\Rightarrow a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n}$$

$$\therefore a_{n+1} + a_n > 0$$

$\therefore a_{n+1} - a_n$ & $a_n - a_{n-1}$ have same sign

i.e.

113

$$a_{n+1} \geq a_n \text{ iff } a_n \geq a_{n-1}$$

$\Rightarrow \{a_n\}$ increasing or decreasing according as

$$a_2 > a_1 \quad \text{or} \quad a_2 < a_1$$

Case I If $a_2 > a_1$, then by mathematical induction it can be proved that $\{a_n\}$ is increasing i.e.

$$a_{n+1} > a_n$$

$$\Rightarrow k + a_n > a_n$$

$$\Rightarrow k + a_n > a_n^2$$

$$\Rightarrow a_n^2 - a_n - k < 0 \quad \rightarrow \textcircled{1}$$

$\because a_n > 0 \quad \forall n$ & $a_n^2 - a_n - k = 0$ has one root +ve. & other -ve.

$\therefore a_n$ is the root of

$$a_n^2 - a_n - k = 0$$

Let the root be $\alpha = \frac{1 + \sqrt{4k+1}}{2}$

other root = $-\frac{k}{\alpha}$

$$\Rightarrow a_n^2 - a_n - k = \alpha (a_n - \alpha) \left(a_n + \frac{k}{\alpha} \right)$$

① \Rightarrow

$$(a_n - \alpha) \left(a_n + \frac{k}{\alpha} \right) < 0$$

$$\therefore a_n + \frac{k}{\alpha} > 0$$

$$\Rightarrow a_n - \alpha < 0 \Rightarrow a_n < \alpha \quad \forall n.$$

Also $a_1 < \alpha$

$\Rightarrow \{a_n\}$ is increasing if $a_1 < \alpha$ &.

$a_n < \alpha \quad \forall n$ i.e. bounded above by α , the root of $x^2 - x - k = 0$

$\Rightarrow \{a_n\}$ is monotonic bounded.

Case II If $a_2 < a_1$, then by mathematical induction $\{a_n\}$ is decreasing i.e.

$$a_{n+1} < a_n$$

$$\Rightarrow k + a_n < a_n^2$$

$$\Rightarrow k + a_n < a_n^2$$

$$\Rightarrow a_n^2 - a_n - k > 0$$

$$\Rightarrow (a_n - \alpha) \left(a_n + \frac{k}{\alpha} \right) > 0$$

$$\therefore a_n + \frac{k}{\alpha} > 0$$

$$\therefore a_n - \alpha > 0 \Rightarrow a_n > \alpha, a_1 > \alpha$$

$\Rightarrow \{a_n\}$ is decreasing if $a_1 > \alpha$ & is bounded below by α

Thus $\{a_n\}$ is monotonic bounded.
sequence and hence cgt.

$$\therefore a_n > 0 \quad \forall n.$$

$$\therefore \lim_{n \rightarrow \infty} a_n \geq 0$$

$$\text{Let } \lim_{n \rightarrow \infty} a_n = l.$$

$$a_{n+1} = \sqrt{k + a_n}$$

$$\Rightarrow a_{n+1}^2 = k + a_n.$$

$$\Rightarrow \left(\lim_{n \rightarrow \infty} a_{n+1} \right)^2 = k + \lim_{n \rightarrow \infty} a_n$$

$$l^2 = k + l$$

$$l^2 - l - k = 0$$

$$\Rightarrow l = \frac{1 \pm \sqrt{4k+1}}{2}$$

$$\because l \geq 0 \quad \therefore l = \frac{1 + \sqrt{4k+1}}{2}$$

the root of equation $x^2 - x - k = 0$

Q# 20 (R.G.B)

Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ $\forall n \in \mathbb{N}$

Show that $\{x_n\}$ is bounded & monotonic. Find the limit

Sol

$$x_{n+1} = 2 - \frac{1}{x_n}$$

$$x_2 = 2 - \frac{1}{x_1} > 1 \quad \because x_1 > 1$$

$$\therefore \frac{1}{x_1} < 1$$

Let $x_k > 1$ for some $k \in \mathbb{N}$

$$\Rightarrow \frac{1}{x_k} < 1$$

$$\Rightarrow -\frac{1}{x_k} > -1$$

$$\Rightarrow 2 - \frac{1}{x_k} > 1$$

$$\Rightarrow x_{k+1} > 1$$

Thus $x_k > 1 \Rightarrow x_{k+1} > 1$

Thus by M. induction.

$$x_n > 1 \quad \forall n$$

$$x_n - x_{n+1} = x_n - \left(2 - \frac{1}{x_n}\right)$$

$$= \frac{x_n^2 - 2x_n + 1}{x_n}$$

$$= \frac{(x_n - 1)^2}{x_n} > 0 \quad \because x_n > 1 \quad \forall n$$

$$\Rightarrow x_n > x_{n+1}$$

$\Rightarrow \{x_n\}$ monotonic bounded and hence convergent

$$\therefore x_n > 1 \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} x_n = l \geq 1$$

$$\lim_{n \rightarrow \infty} x_{n+1} = \frac{117}{2} - \frac{1}{\lim_{n \rightarrow \infty} x_n}$$

$$l = 2 - \frac{1}{l}$$

$$l^2 = 2l - 1$$

$$l^2 - 2l + 1 = 0$$

$$(l-1)^2 = 0 \Rightarrow \boxed{l=1}$$

Q: 21 ^{Crashit}

$$\text{If } x_1 = 4 \quad x_{n+1} = 3 - \frac{2}{x_n}$$

$$x_2 = 3 - \frac{2}{x_1} = 3 - \frac{2}{4}$$

$$= 3 - \frac{1}{2} = \frac{5}{2} = 2.5 > 2$$

$$x_3 = 3 - \frac{2}{x_2} = 3 - \frac{4}{5} = \frac{11}{5} = 2.2 > 2$$

Let $x_k > 2$ for some $k \in \mathbb{N}, k \geq 2$

$$+ \frac{1}{x_k} < \frac{1}{2}$$

$$\Rightarrow \frac{2}{x_k} < 1$$

$$\Rightarrow -\frac{2}{x_k} > -1$$

$$\Rightarrow 3 - \frac{2}{x_k} > 2$$

$$\Rightarrow x_{k+1} > 2$$

$$\text{Thus } x_k > 2$$

$$\Rightarrow x_{k+1} > 2$$

$$\Rightarrow x_n > 2 \quad \forall n \geq 1$$

118

$$x_n - x_{n+1} = x_n - \left(3 - \frac{2}{x_n}\right)$$

$$= \frac{x_n^2 - 3x_n + 2}{x_n}$$

$$= \frac{(x_n - 2)(x_n - 1)}{x_n} > 0$$

$$\therefore x_n > 2 \quad \therefore x_n - 2 > 0 \quad \begin{matrix} x_n - 1 > 0 \\ x_n > 0 \end{matrix}$$

$$\Rightarrow x_n > x_{n+1}$$

$\Rightarrow \{x_n\}$ is decreasing & bounded and hence
Convergent. $\therefore \lim_{n \rightarrow \infty} a_n \geq 2$.

Let $\lim_{n \rightarrow \infty} a_n = l$.

$$x_{n+1} = 3 - \frac{2}{x_n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_{n+1} = 3 - \frac{2}{\lim_{n \rightarrow \infty} x_n}$$

$$l = 3 - \frac{2}{l}$$

$$l^2 = 3l - 2$$

$$l^2 - 3l + 2 = 0$$

$$(l - 2)(l - 1) = 0$$

$$\Rightarrow l = 2, \text{ or } l = 1$$

$$\text{But } l \geq 2 \Rightarrow l = 2$$

Q: 22 (Generalisation of 119 Q20 & Q21 made myself)
Define a sequence as

$a \geq 2, x_1 > a-1; x_{n+1} = a - \frac{a-1}{x_n}$
Prove that $\{x_n\}$ converges to a limit. Find the limit

Sol

$$x_1 > a-1$$

$$x_2 = a - \frac{a-1}{x_1} \rightarrow \textcircled{1}$$

$$\therefore x_1 > a-1$$

$$1 > \frac{a-1}{x_1}$$

$$\frac{a-1}{x_1} < 1$$

$$-\frac{a-1}{x_1} > -1$$

$$\Rightarrow a - \frac{a-1}{x_1} > a-1$$

$$\Rightarrow x_2 > a-1$$

Let $x_k > a-1$ for some $k \geq 2$.

$$\Rightarrow \frac{1}{x_k} < \frac{1}{a-1}$$

$$\Rightarrow \frac{a-1}{x_k} < 1$$

$$\Rightarrow -\frac{a-1}{x_k} > -1$$

$$\Rightarrow a - \frac{a-1}{x_n} \stackrel{120}{>} a-1$$

$$\Rightarrow x_{n+1} > a-1$$

$$\text{Hence } x_n > a-1 \Rightarrow x_{n+1} > a-1$$

$$\text{Thus } x_n > a-1 \quad \forall n.$$

$$x_n - x_{n+1} = x_n - \left(a - \frac{a-1}{x_n} \right)$$

$$= \frac{x_n^2 - ax_n + (a-1)}{x_n}$$

$$= \frac{x_n^2 - (a-1)x_n - x_n + (a-1)}{x_n}$$

$$\Rightarrow \frac{x_n (x_n - (a-1)) - 1 (x_n - (a-1))}{x_n}$$

$$\Rightarrow \frac{(x_n - 1)(x_n - (a-1))}{x_n} \geq 0$$

$$\because x_n > a-1 \geq 1 \quad \therefore a \geq 2$$

$$\Rightarrow x_n \geq x_{n+1}$$

$\Rightarrow \{x_n\}$ is monotone decreasing and bounded & hence convergent

$$\text{Also } x_n > a-1$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_n \geq a-1$$

K 2 1 1 1

Let $\lim_{n \rightarrow \infty} x_n = l$ (21)

Now $x_{n+1} = a - \frac{a-1}{x_n}$
 $\lim_{n \rightarrow \infty} x_{n+1} = a - \frac{a-1}{\lim_{n \rightarrow \infty} x_n}$

$$l = a - \frac{a-1}{l}$$

$$l^2 = la - (a-1)$$

$$l^2 - la + a - 1 = 0$$

$$l^2 - (a-1)l - l + a - 1 = 0$$

$$l[l - (a-1)] - 1[l - (a-1)] = 0$$

$$\Rightarrow [l - (a-1)][l - 1] = 0$$

$$\Rightarrow l - (a-1) = 0 \quad | \quad l - 1 = 0$$

$$l = a-1$$

$$\text{But } l \neq a-1 \Rightarrow l = 1$$

Q#23 (R.G.B exercise 3.3)

Let $x_1 \geq 2$ & $x_{n+1} = 1 + \sqrt{x_n - 1}$. $\forall n \in \mathbb{N}$
 Show that $\{x_n\}$ is decreasing and bounded below by 2. Find the limit.

Sol

$$x_1 \geq 2$$

$$x_2 = 1 + \sqrt{x_1 - 1} \geq 2 \quad \because x_1 \geq 2$$

494
WARRIN

122

Let $x_k \geq 2$ for $\forall k \in \mathbb{N}$

$$\Rightarrow x_k - 1 \geq 1$$

$$\Rightarrow |x_k - 1| \geq 1$$

$$\Rightarrow 1 + \sqrt{x_k - 1} \geq 2$$

$$\Rightarrow x_{k+1} \geq 2$$

Thus $x_n \geq 2 \quad \forall n$.

$$x_n - x_{n+1} = x_n - 1 - \sqrt{x_n - 1}$$

$$= (\overline{x_n - 1})^2 - \sqrt{x_n - 1}$$

$$= \sqrt{x_n - 1} (\sqrt{x_n - 1} - 1) \geq 0 \quad \because x_n \geq 2$$

$$\Rightarrow x_n \geq x_{n+1}$$

$\Rightarrow \{x_n\}$ is monotone decreasing & bounded.
and hence convergent.

$$\text{Also } x_n \geq 2$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_n \geq 2$$

$$\text{Let } \lim_{n \rightarrow \infty} x_n = l$$

$$x_{n+1} = 1 + \sqrt{x_n - 1}$$

$$\lim_{n \rightarrow \infty} x_{n+1} = 1 + \sqrt{\lim_{n \rightarrow \infty} x_n - 1}$$
$$l = 1 + \sqrt{l - 1}$$

$$\Rightarrow l-1 = \sqrt[l-1]{l-1}$$

$$(l-1)^2 = l-1$$

$$\Rightarrow (l-1)^2 - (l-1) = 0$$

$$\Rightarrow (l-1)[l-1-1] = 0$$

$$(l-1)(l-2) = 0$$

$$\Rightarrow l=1 \quad l=2$$

$$\text{Thus } l=2 \quad \because l \geq 2$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 2$$

Q# 24 (Generalisation of Q23 made by myself.)

Let $x_1 \geq a^2+1$, $a \geq 1$ & $x_{n+1} = a^2 + \sqrt{x_n - a^2}$
 $\forall n \in \mathbb{N}$. Show that $\{x_n\}$ decreasing and
 bounded below a^2+1 . Find the limit.

Sol $x_1 \geq a^2+1$

$$x_2 = a^2 + \sqrt{x_1 - a^2} \geq a^2+1 \because x_1 \geq a^2+1$$

Let $x_k \geq a^2+1$ for some $k \in \mathbb{N}$

$$\Rightarrow x_k - a^2 \geq 1 \Rightarrow \sqrt{x_k - a^2} \geq 1$$

$$\Rightarrow a^2 + \sqrt{x_k - a^2} \geq a^2+1$$

$$\Rightarrow x_{k+1} \geq a^2+1$$

$$\text{Thus } x_n \geq a^2+1$$

124

$$\begin{aligned}
 x_n - x_{n+1} &= x_n - (a^2 + \sqrt{x_n - a^2}) \\
 &= x_n - a^2 - \sqrt{x_n - a^2} \\
 &= \overline{x_n - a^2} (\sqrt{x_n - a^2} - 1) \geq 0
 \end{aligned}$$

$$\therefore x_n \geq a^2 + 1$$

$$\Rightarrow x_n \geq x_{n+1}$$

$\Rightarrow \{x_n\}$ is monotone decreasing and bounded and hence convergent

$$\therefore x_n \geq a^2 + 1$$

$$\therefore \lim_{n \rightarrow \infty} x_n \geq a^2 + 1$$

$$\text{Let } \lim_{n \rightarrow \infty} x_n = l$$

$$\begin{aligned}
 x_{n+1} &= a^2 + \sqrt{x_n - a^2} \\
 \lim_{n \rightarrow \infty} x_{n+1} &= a^2 + \sqrt{\lim_{n \rightarrow \infty} x_n - a^2}
 \end{aligned}$$

$$l = a^2 + \sqrt{l - a^2}$$

$$l - a^2 = \sqrt{l - a^2}$$

$$(l - a^2)^2 = l - a^2$$

$$(l - a^2)^2 - (l - a^2) = 0$$

$$(l - a^2)(l - a^2 - 1) = 0$$

$$\begin{aligned}
 l &= a^2 \\
 \text{or } l &= a^2 + 1
 \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} x_n = a^2 + 1$

Q:25. A sequence ¹²⁵ $\{x_n\}$ is defined as

$$x_1 = a > 0 \quad x_{n+1} = \frac{\sqrt{ab^2 + x_n^2}}{a+1}$$

$b > a \quad n \geq 1$

Show that $\{x_n\}$ is cgt and find its limit

Sol

It is given that

$$a < b \Rightarrow x_1 < b \rightarrow \textcircled{1}$$

$$x_{n+1}^2 - b^2 = \frac{ab^2 + x_n^2}{a+1} - b^2$$

$$= \frac{x_n^2 - b^2}{a+1} < 0 \text{ whenever } x_n^2 - b^2 < 0$$

$$\Rightarrow x_{n+1}^2 - b^2 < 0 \text{ when } x_n^2 - b^2 < 0$$

$$\Rightarrow x_{n+1} < b \text{ when } x_n < b \rightarrow \textcircled{2}$$

By ① & ②

By mathematical induction, we have

$$x_n < b \quad \forall n$$

OR

$$x_1 < b \Rightarrow x_n < b \text{ for } n=1$$

$$\text{Let } x_k < b \text{ for } k \geq 1$$

$$\Rightarrow x_k^2 - b^2 < 0$$

$$\Rightarrow x_k^2 < b^2$$

$$\Rightarrow ab^2 + x_k^2 < \overset{126}{b^2 + ab^2}$$

$$\Rightarrow \frac{ab^2 + x_k^2}{a+1} < \frac{b^2(a+1)}{a+1} \quad \because a+1 > 0$$

$$\Rightarrow \sqrt{\frac{ab^2 + x_k^2}{a+1}} < b$$

$$\Rightarrow x_{k+1} < b$$

Thus by mathematical induction

$$x_n < b \quad \forall n$$

$\Rightarrow \{x_n\}$ is bounded above.

Again

$$\begin{aligned} x_{n+1}^2 - x_n^2 &= \frac{ab^2 + x_n^2}{a+1} - x_n^2 \\ &= \frac{a(b^2 - x_n^2)}{a+1} > 0 \quad \left[\because x_n < b \right] \end{aligned}$$

$$\Rightarrow x_{n+1}^2 > x_n^2$$

$$\Rightarrow |x_{n+1}| > |x_n|$$

$$\Rightarrow |x_{n+1}| > |x_n|$$

$$\Rightarrow x_{n+1} > x_n \quad \left[\because x_n > 0 \right]$$

$\Rightarrow \{x_n\}$ is increasing
Thus $\{x_n\}$ is monotonically increasing and bounded above

note $|a| > |b| \nRightarrow a > b$
but if $a > 0, b > 0$, then

$$|a| > |b| \Rightarrow a > b$$

Also $|a| < |b| \nRightarrow a < b$
if $a > 0, b > 0$, then

$$|a| < |b| \Rightarrow a < b$$

and hence convergent. ¹²⁷

$$\text{Let } \lim_{n \rightarrow \infty} x_n = l$$

$$\text{Now } x_{n+1}^2 = \frac{ab^2 + x_n^2}{a+1}$$

$$\Rightarrow \left(\lim_{n \rightarrow \infty} x_{n+1} \right)^2 = \frac{ab^2 + \left(\lim_{n \rightarrow \infty} x_n \right)^2}{a+1}$$

$$l^2 = \frac{ab^2 + l^2}{a+1}$$

$$al^2 + l^2 = ab^2 + l^2$$

$$\Rightarrow l^2 = b^2 \Rightarrow l = \pm b$$

$$\therefore x_n > 0 \quad \forall n.$$

$$\therefore \lim_{n \rightarrow \infty} x_n \geq 0 \quad \forall n.$$

Hence sequence $\{x_n\}$ converges to b

Q# 26 If a_1, b_1 are two true unequal numbers and a_n, b_n are defined as

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) \quad b_n = \sqrt{a_{n-1}b_{n-1}} \quad n \geq 2.$$

prove that two sequences $\{a_n\}$ and $\{b_n\}$ are monotonic, one increasing and the other decreasing and that they tend to the same limit.

Sol Let $a_1 > b_1$.

Since for any two true numbers, the A.M is greater than the G.M.

12.8

$$\therefore a_n > b_n$$

$$\text{Also } a_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(a_n + a_n) = a_n$$

$$\therefore b_n < a_n$$

$\Rightarrow \{a_n\}$ is \downarrow

$$\Rightarrow a_1 > a_2 > a_3 > a_4 > a_5 > \dots$$

$$\text{Again } b_{n+1} = \sqrt{a_n \cdot b_n} > \sqrt{b_n \cdot b_n} = b_n \quad [a_n > b_n]$$

$\therefore \{b_n\}$ is monotone decreasing

$$\text{Now } a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) > \frac{1}{2}(b_{n-1} + b_{n-1}) = b_{n-1}$$

$$\therefore a_n > b_n$$

$$\Rightarrow a_n > b_{n-1} > b_{n-2} > \dots > b_2 > b_1$$

$$\Rightarrow a_n > b_1 \quad \forall n$$

$\Rightarrow \{a_n\}$ is \downarrow and bounded. & hence cgt.

$$\text{Again } b_n = \sqrt{a_{n-1} \cdot b_{n-1}} < \sqrt{a_{n-1} \cdot a_{n-1}} = a_{n-1} \text{ by (i)}$$

$$\text{i.e. } b_n < a_{n-1} < a_{n-2} < \dots < a_2 < a_1$$

$\therefore \{b_n\}$ is bounded above and being monotone increasing is convergent

$$\text{let } \lim_{n \rightarrow \infty} a_n = l_1 \quad \& \quad \lim_{n \rightarrow \infty} b_n = l_2$$

$$\text{Since } a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$$

$$2a_n = a_{n-1} + b_{n-1}$$

$$\therefore 2a_{n+1} = \frac{129}{a_n + b_n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} 2a_{n+1} = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$2l_1 = l_1 + l_2$$

$$\Rightarrow l_1 = l_2$$

$\Rightarrow \{a_n\}$ & $\{b_n\}$ converge to the same limit.

Q# 27 If $a_1 > 0, b_1 > 0$ and $a_n = \sqrt{a_{n-1} \cdot b_{n-1}}$
and $b_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1} + b_{n-1}}$, prove that

(i) $\{a_n\}$ and $\{b_n\}$ are monotonic, the one increasing and the other decreasing (ii)

$\{a_n\}$ and $\{b_n\}$ both converge to the same limit. $\rightarrow \textcircled{1}$

Sol Suppose $a_1 > b_1$.

\therefore for any two true nos G.M $>$ H.M.

$$a_n > b_n \quad \forall n \quad \rightarrow \textcircled{2}$$

$$\text{Also } a_{n+1} = \sqrt{a_n \cdot b_n} < \sqrt{a_n \cdot a_n} = a_n$$

$\Rightarrow \{a_n\}$ is \searrow i.e.

$$a_1 > a_2 > a_3 > a_4 \dots > a_n >$$

$$\Rightarrow a_1 \geq a_n \quad \forall n \quad \rightarrow \textcircled{3}$$

$$\text{Again } b_{n+1} = \frac{2a_nb_n}{a_n + b_n} > \frac{2b_n \cdot b_n}{b_n + b_n} = b_n \quad [a_n > b_n]$$

$$b_{n+1} > b_n$$

$\{b_n\}$ is monotone increasing $b_1 < b_2 < b_3 \dots < b_n$

$$\text{or } b_n \geq b_1 \quad \forall n \quad \rightarrow \textcircled{4}$$

is greater than ...

From ①, ②, ③ & ¹³⁶④ we get

$$a_1 \geq a_n > b_n \geq b_1 \quad \forall n \rightarrow \textcircled{5}$$

Now since $a_n > b_1 \quad \forall n$

$\Rightarrow \{a_n\}$ is bounded below and being \downarrow converges to a limit.

Again $\textcircled{5} \Rightarrow b_n < a_1 \quad \forall n$

$\Rightarrow \{b_n\}$ is bounded above and being \uparrow is convergent.

$$\text{Let } \lim_{n \rightarrow \infty} a_n = a \quad \& \quad \lim_{n \rightarrow \infty} b_n = b$$

$$\text{Now } a_{n+1} = \sqrt{a_n \cdot b_n}$$

$$\text{or } a_{n+1}^2 = a_n b_n$$

$$\therefore \text{ we have } \lim_{n \rightarrow \infty} a_{n+1}^2 = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$a^2 = ab$$

$\Rightarrow a = b$ which proves the result.

Q. 28 Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by $x_1 = \frac{1}{2}$ & $y_1 = 1$.

$$\text{and } x_n = \sqrt{x_{n-1} y_{n-1}} \quad n = 2, 3, \dots$$

$$\frac{1}{y_n} = \frac{1}{2} \left(\frac{1}{x_n} + \frac{1}{y_{n-1}} \right) \quad n = 2, 3, 4, \dots$$

prove that $x_{n-1} < x_n < y_n < y_{n-1} \quad n = 2, 3, \dots$
and deduce that both sequences converge to the same limit l where $\frac{1}{2} < l < 1$

Sol If $a < x < b$, then G.M. = \sqrt{ab}

$$\text{and } H = \left[\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right) \right]^{-1} = \frac{2ab}{a+b}$$

$$\text{Also } a < H < G < b$$

we are given that 131

$$\frac{1}{2} = x_1 < y_1 = 1$$

$$\Rightarrow \text{let } x_{n-1} < y_{n-1}$$

$$x_{n-1} < x_n < y_{n-1}$$

$$[\because x_n = \sqrt{x_{n-1} y_{n-1}}]$$

$$\text{Further } x_n < y_n < y_{n-1}$$

because y_n is H.M of x_n & y_{n-1} .

$$\Rightarrow x_{n-1} < x_n < y_n < y_{n-1} \quad n=2, 3, \dots$$

$\Rightarrow \{x_n\}$ \uparrow and is bounded above by $y_1 = 1$

The sequence $\{y_n\}$ decreases and is bounded below by $x_1 = \frac{1}{2}$. Hence both the sequence converge.

Suppose $x_n \rightarrow l$ as $n \rightarrow \infty$ and $y_n \rightarrow m$ as $n \rightarrow \infty$

$$\text{Then } l^2 = lm \Rightarrow \boxed{l = m}$$

$$\frac{1}{m} = \frac{1}{2} \left(\frac{l+m}{lm} \right) \Rightarrow \boxed{l = m}$$

Q 29 If a sequence $\{a_n\}$ is defined by

$$a_{n+1} = 1 + \frac{1}{a_n} \quad \forall n \geq 2, a_1 > 0, d_1 = 1$$

prove that the sequence $\{a_n\}$ is cgt and $a_2 = 2$

$$\lim_{n \rightarrow \infty} a_n = \frac{1 + \sqrt{5}}{2}$$

$$\underline{\text{Sol}} \quad a_3 \leq 1 + \frac{1}{2} < 2$$

$$a_3 = \frac{3}{2}$$

$$a_4 = 1 + \frac{1}{a_3} = 1 + \frac{2}{3} < 2$$

$$\text{Let } a_{n+1} < 2 \quad \text{for } n \geq 2$$

is greater than

$\Rightarrow \{a_{2k}\}$ is ¹³³ monotone decreasing sequence
Abo

$$1 \leq a_{2k-1} \leq 2 \quad \forall k \in \mathbb{N}$$

$$1 \leq a_{2k} \leq 2 \quad \forall k \in \mathbb{N}$$

$\Rightarrow \{a_{2k}\}$ and $\{a_{2k-1}\}$ are bounded monotone sequence and hence cgt.

$$\text{Let } l_1 = \lim_{k \rightarrow \infty} a_{2k-1}$$

$$\text{Now } l_2 = \lim_{k \rightarrow \infty} a_{2k}$$

$$a_n = 1 + \frac{a_{n-2}}{1+a_{n-2}} \quad \text{for } n \geq 3$$

$$a_{2k-1} = 1 + \frac{a_{2k-3}}{1+a_{2k-3}}$$

$$a_{2k} = 1 + \frac{a_{2k-2}}{1+a_{2k-2}}$$

$$\lim_{k \rightarrow \infty} a_{2k-1} = 1 + \frac{\lim_{k \rightarrow \infty} a_{2k-3}}{1 + \lim_{k \rightarrow \infty} a_{2k-3}}$$

$$l_1 = 1 + \frac{l_1}{1+l_1}$$

$$\lim_{k \rightarrow \infty} a_{2k} = 1 + \frac{\lim_{k \rightarrow \infty} a_{2k-2}}{1 + \lim_{k \rightarrow \infty} a_{2k-2}}$$

$$l_2 = 1 + \frac{l_2}{1+l_2}$$

Thus l_1 & l_2 both satisfy equation $l^2 - l - 1 = 0$

134

$$l = \frac{1 \pm \sqrt{5}}{2}$$

But $1 \leq a_n \leq 2 \quad \forall n$
 So $l_1, l_2 > 0$

$$\Rightarrow l_1 = \frac{1 + \sqrt{5}}{2} = l_2$$

Q #30 (Grashil) If a sequence $\{a_n\}$ is defined by

$$a_2 > a_1 > 0, \quad a_{n+1} = \frac{a_n + a_{n-1}}{2} \quad \forall n \geq 2$$

$$\text{or } a_n = \frac{a_{n-1} + a_{n-2}}{2} \quad \forall n \geq 2$$

prove that the sequence converges also find Limit

Sol # $a_{n+2} - a_n = \frac{1}{2} [a_{n+1} + a_n] - a_n$
 $= \frac{1}{2} [a_n - a_{n-2}] \rightarrow \text{①}$

$$\therefore a_1 < a_2$$

putting $n = 3, 4, 5, \dots$ in relation

$$a_n = \frac{1}{2} [a_{n-1} + a_{n-2}]$$

$$a_3 = \frac{1}{2} [a_2 + a_1]$$

$$a_4 = \frac{1}{2} [a_3 + a_2]$$

$$a_5 = \frac{1}{2} [a_4 + a_3]$$

135

$$a_m = \frac{1}{2} [a_{m-1} + a_{m-2}]$$

$$\therefore a_1 < a_2$$

$$a_3 = \frac{1}{2} [a_1 + a_2] < \frac{1}{2} [a_2 + a_2] = a_2$$

$$\text{But } a_3 = \frac{1}{2} [a_1 + a_2]$$

$$\Rightarrow a_1 < a_3 < a_2$$

$$a_3 < a_4 < a_2$$

$$a_3 < a_5 < a_4$$

$$a_5 < a_6 < a_4$$

Thus it appears.

$$a_1 < a_3 < a_5 \dots$$

$$a_2 > a_4 > a_6 \dots$$

from ① let $n = 2m$

$$a_{2m+2} - a_{2m} = \frac{1}{2} [a_{2m+1} - a_{2m}]$$

$$\therefore a_{2m+2} < a_{2m} \Rightarrow a_{2m+2} - a_{2m} < 0$$

$$\Rightarrow a_{2m+1} - a_{2m} < 0$$

$$\Rightarrow a_{2m+1} < a_{2m}$$

but

$$\Rightarrow a_{2m} < a_{2m-2} < \dots < a_4 < a_2$$

\therefore every odd term is less than every even term
ie $\{a_{2m+1}\}$ is increasing and bounded above
by a_2 and is therefore convergent

Similarly even term ¹³⁶ subsequence $\{a_{2n}\}$ is convergent. we show that both converge to same limit

Let $\lim_{n \rightarrow \infty} a_{2n+1} = l_1$ & $\lim_{n \rightarrow \infty} a_{2n} = l_2$
from recursion relation.

$$a_{2n} = \frac{1}{2} [a_{2n-1} + a_{2n-2}]$$

$$\lim_{n \rightarrow \infty} a_{2n} = \frac{1}{2} [\lim_{n \rightarrow \infty} a_{2n-1} + \lim_{n \rightarrow \infty} a_{2n-2}]$$

$$l_2 = \frac{1}{2} [l_1 + l_2]$$

$$2l_2 = l_1 + l_2$$

$$l_1 = l_2$$

Thus $\{a_n\}$ is cgt.

Now from $a_n = \frac{1}{2} [a_{n-1} + a_{n-2}]$

$$a_{n+1} = \frac{1}{2} [a_n + a_{n-1}]$$

$$a_3 = \frac{1}{2} [a_2 + a_1]$$

$$a_4 = \frac{1}{2} [a_3 + a_2]$$

$$a_5 = \frac{1}{2} [a_4 + a_3]$$

$$\vdots$$
$$a_k = \frac{1}{2} [a_{k-1} + a_{k-2}]$$

$$a_{k+1} = \frac{1}{2} [a_k + a_{k-1}]$$

137

$$- - a_{k-1} + a_{k-2} + a_k + a_{k-1}]$$

$$+ \dots - 2a_{k-1} + a_k.]$$

$$\Rightarrow \frac{1}{2} a_k + a_{k+1} = \frac{1}{2} [a_1 + 2a_2]$$

Takij Limit when $k \rightarrow \infty$

$$\frac{1}{2}l + l = \frac{1}{2} [a_1 + 2a_2]$$

$$\frac{3}{2}l = \frac{1}{2} [a_1 + 2a_2]$$

$$l_2 = \frac{1}{3} [a_1 + 2a_2]$$

Q31 If a sequence a_n is defined

$$\text{by } a_n = \frac{a}{1 + a_{n-1}} \quad a_{70} \quad a_{1,70} \quad \text{Hinz. 2}$$

Then show that the sequence converges to the true root of $x^2 + x - a = 0$

Sol

$$a_n = \frac{a}{1 + a_{n-1}} \rightarrow \textcircled{1}$$

$$a_n - a_{n-2} = \frac{9}{1+a_{n-1}} + \frac{9}{1+a_{n-3}}$$

138

$$= q \left[\frac{1+a_{n-3} - 1-a_{n-1}}{(1+a_{n-1})(1+a_{n-3})} \right]$$

$$= \frac{-a(a_{n-1} - a_{n-3})}{(1+a_{n-1})(1+a_{n-3})} \rightarrow (2)$$

$$= \frac{-a \left[\frac{1+a}{1+a_{n-2}} - \frac{q}{1+a_{n-4}} \right]}{(1+a_{n-1})(1+a_{n-3})}$$

$$= \frac{a^2(a_{n-2} - a_{n-4})}{(1+a_{n-1})(1+a_{n-3})(1+a_{n-2})(1+a_{n-4})} \rightarrow (3)$$

(3) \Rightarrow that $a_n - a_{n-2}$ & $a_{n-2} - a_{n-4}$ have same sign. So even and odd terms form separate monotone sub-sequences.

From (2) we note that if odd numbered sub-sequence form a monotone decreasing sub-sequence, then sequence of even no terms form a sub-sequence of increasing terms and vice versa.

Since every term ¹³⁹ of the sequence is +ve.

$$a - a_n = a - \frac{a}{1+a_{n-1}} = \frac{a a_{n-1}}{1+a_{n-1}} > 0$$

$$\Rightarrow a - a_n > 0 \quad \forall n$$

$$\Rightarrow a_n < a \quad \forall n$$

Thus $0 < a_n < a$.

OR
$$a_n = \frac{a}{1+a_{n-1}} < a \quad \because a_{n-1} > 0$$

Thus monotone increasing sub-sequence is bounded above by a and the monotone decreasing subsequence is bounded below by 0 .

Hence the two sub-sequences converge.

Let even no terms converges to l_1 & odd no term converge to l_2 .

$$\text{Then } \lim_{n \rightarrow \infty} a_n = \frac{a}{1 + \lim_{n \rightarrow \infty} a_{n-1}}$$

(a) we get for n even $l_1 = \frac{a}{1+l_2}$ or $l_1 l_2 + l_1 = a$.

(b) for n odd $l_2 = \frac{a}{1+l_1}$ or $l_1 l_2 + l_2 = a$.

$$\Rightarrow l_1 l_2 + l_1 = l_1 l_2 + l_2$$

$$\Rightarrow \boxed{l_1 = l_2}$$

Q# 32

140

A sequence $\{a_n\}$ is defined as $a_1 = 1$
 $a_{n+1} = \frac{4+3a_n}{3+2a_n} \quad n \geq 1$ Show that $\{a_n\}$

Converges and Find Limit

Sol $a_1 = 1 \quad a_2 = \frac{4+3a_1}{3+2a_1} = \frac{7}{5} > 1$

$$a_1 > a_2$$

let $a_{n+1} > a_n$

$$\text{Then } a_{n+2} - a_{n+1} = \frac{4+3a_{n+1}}{3+2a_{n+1}} - \frac{4+3a_n}{3+2a_n}$$

$$= \frac{a_{n+1} - a_n}{(3+2a_{n+1})(3+2a_n)} > 0$$

$$[\because a_{n+1} > a_n \\ a_n > 0 \quad \forall n]$$

$$a_{n+2} - a_{n+1} > 0$$

$$\Rightarrow a_{n+2} > a_{n+1}$$

By mathematical induction $\{a_n\}$ is increasing

$$a_{n+1} = \frac{4+3a_n}{3+2a_n} = \frac{3}{2} - \frac{1}{2(3+2a_n)}$$

$$= \frac{3}{2} - (\text{a true quantity less than } 1) \quad (a_2 > a_1 = 1)$$

$$< \frac{3}{2} \quad \Rightarrow a_{n+1} < \frac{3}{2} \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded above.

Thus $\{a_n\}$ is bounded ¹⁴¹ monotonic and hence convergent. Let $\lim_{n \rightarrow \infty} a_n = l$.

Then

$$a_{n+1} = \frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow l = \frac{4+3l}{3+2l}$$

$$3l+2l^2 = 4+3l$$

$$\Rightarrow 2l^2 = 4 \Rightarrow l^2 = \frac{4}{2} = 2$$

$$\Rightarrow l = \pm\sqrt{2}$$

But l cannot be $-\sqrt{2}$. $l = \sqrt{2}$

Q: 33 # A sequence is defined by

$$a_0 = 1 \quad a_1 = \frac{3}{2} \quad \dots \quad a_{n+1} = 1 + \frac{1}{2}a_n$$

Show that the sequence converges by showing that the sequence is bounded and find its

Limit

$$a_0 = 1 \quad a_1 = \frac{3}{2} = 1.5$$

Sol

$$a_2 = \frac{7}{4} = 1.75$$

$$a_3 = 1 + \frac{1}{2}a_2 = 1 + \frac{1}{2} \cdot \frac{7}{4} = \frac{15}{8} = 1.88$$

The sequence appears to increase.
Hence it is possible only if

$$a_{n+1} > a_n$$

$$\Rightarrow 1 + \frac{1}{2}a_n > a_n$$

$$1 > \frac{1}{2}a_n$$

$$\Rightarrow 2 > a_n$$

$$a_n < 2 \quad \forall n$$

142

\Rightarrow Sequence is bounded above by 2.
Hence the sequence is bounded and thus
convergent

$$\text{Let } \lim_{n \rightarrow \infty} a_n = l.$$

$$\Rightarrow l = 1 + \frac{1}{2}l \Rightarrow l = 2.$$

Thus $\lim_{n \rightarrow \infty} a_n = 2$

Theorem If $|a| < 1$, then $\lim_{n \rightarrow \infty} a^n = 0$

Proof Case I Let $0 < a < 1$

Then $\{a^n\}$ is a bounded monotone sequence and thus convergent.

$$\text{Let } \lim_{n \rightarrow \infty} a^n = l.$$

$$\text{Then } l = \lim_{n \rightarrow \infty} a^{n+1} = \lim_{n \rightarrow \infty} a a^n$$

$$l = al$$

$$\Rightarrow l(1-a) = 0 \Rightarrow l = 0 \text{ or } a = 1$$

If $l \neq 0$, then $a = 1$ which is impossible.

because $|a| < 1 \Rightarrow -1 < a < 1$

Therefore $\lim_{n \rightarrow \infty} a^n = 0$ if $0 < a < 1$

Corollary II Let $-1 < a < 0$

Then $0 < -a < 1$

Now

$$(-a)^n = (-1)^n a^n$$

$\{(-1)^n\}$ is bounded & $\lim_{n \rightarrow \infty} a^n = 0$

$$\Rightarrow \lim_{n \rightarrow \infty} (-1)^n a^n = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} (-a)^n = 0$$

$$\begin{aligned} \lim_{n \rightarrow \infty} a^n &= \lim_{n \rightarrow \infty} [(-1)(-a)]^n \\ &= \lim_{n \rightarrow \infty} (-1)^n (-a)^n = 0 \end{aligned}$$

Case III If $a = 0$, then $\lim_{n \rightarrow \infty} a^n = 0$

If $a = 0$, then OR $\lim_{n \rightarrow \infty} a^n = 0$

Let us assume that $0 < |a| < 1$

Let $\epsilon > 0$

$$|a^n - 0| < \epsilon$$

$$|a^n| < \epsilon \Rightarrow |a|^n < \epsilon$$

$$n \ln |a| < \ln \epsilon$$

$$n > \frac{\ln \epsilon}{\ln |a|}$$

$$\therefore \ln |a| < 0$$

because $0 < |a| < 1$

This inequality provides us a clue to choose n , (fixed) - Let us consider

Cor I If $\epsilon < 1$

Then $\lim_{n \rightarrow \infty} a^n < 0$ and let $n_1 > \frac{\ln \epsilon}{\ln |a|}$
 such that
 $|a^n - 0| < \epsilon \quad \forall n > n_1$
 $\Rightarrow \lim_{n \rightarrow \infty} a^n = 0$

Case II If $\epsilon < 1$, then $\lim_{n \rightarrow \infty} a^n < 0$
 and $\frac{\ln \epsilon}{\ln |a|} \leq 0$

In this case we can choose an $n_1 +ve$
 and will have

$|a^n - 0| < \epsilon \quad \forall n > n_1$
 $\Rightarrow \lim_{n \rightarrow \infty} a^n = 0$

Theorem If $a > 0$, then $\lim_{n \rightarrow \infty} a^{1/n} = 1$

Proof # Case-I let $a > 1$

Then $a^{1/n} > 1$

$\{a^{1/n}\}$ is bounded below by 1.

Also $a > a^{1/2} > a^{1/3} > a^{1/4} > \dots$

$\Rightarrow \{a^{1/n}\}$ is bounded and decreasing
 and hence convergent

Let $\lim_{n \rightarrow \infty} a^{1/n} = l$

$\therefore a^{1/n} > 1$

$\Rightarrow \lim_{n \rightarrow \infty} a^{1/n} > 1$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{2}{n} a^{1/n} = \lim_{n \rightarrow \infty} (a^{1/n})^2 = l^2$$

The sub-sequence $\{a^{2/n}\}$ & $\{a^{1/n}\}$ converge to same limit l^2 hence.

$$l = l^2$$

$$l(l-1) = 0$$

$$l = 0 \text{ or } l = 1$$

Since $l \neq 0$, $l = 1$

$$\Rightarrow \lim_{n \rightarrow \infty} a^{1/n} = 1$$

Let $a > 1$, OR then $a^{1/n} > 1 \quad \forall n$

Let $a^{1/n} = 1 + h_n$ where $h_n > 0$

$$a = (1 + h_n)^n = 1 + n h_n + \frac{n(n-1)}{2!} h_n^2 + \dots$$

$$\geq 1 + n h_n$$

$$\Rightarrow \frac{a-1}{n} \geq h_n$$

$$\Rightarrow 0 < h_n \leq \frac{a-1}{n} \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{a-1}{n} = 0$$

By squeeze principle

$$\lim_{n \rightarrow \infty} h_n = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} a^{1/n} = \lim_{n \rightarrow \infty} (1 + h_n) = 1 + 0 = 1$$

Case

if $a < 1$

145

If $a \geq 1$
then $\lim_{n \rightarrow \infty} a^{1/n} = 1$

Case II If $0 < a < 1$

Then $\frac{1}{a} > 1$ and by case I above

$$\lim_{n \rightarrow \infty} \left(\frac{1}{a}\right)^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{a^{1/n}} = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} a^{1/n} = 1$$

Theorem # Discuss the nature of the sequence $\{a^n\}$ for $a \in \mathbb{R}$.

Sol The behaviour of the sequence $\{a^n\}$ depends upon the value of a

Case I Let $a > 1$

Then $a \geq 1+h$ where $h > 0$

$$a^n \geq (1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \dots + h^n$$

$$a^n \geq 1 + nh$$

$$\text{as } n \rightarrow \infty, 1 + nh \rightarrow \infty$$

$$\Rightarrow a^n \rightarrow \infty \text{ as } n \rightarrow \infty$$

$\Rightarrow \{a^n\}$ diverges

Case II Let $a \geq 1$

$$a^n = 1$$

$\Rightarrow \{a^n\}$ is a Constant sequence and Converges

Case III

Let $0 < a < 1$

Then $\frac{1}{a} > 1$

$\frac{1}{a} = 1+h$ for some $h > 0$

$$\frac{1}{a^n} = (1+h)^n > nh$$

$$\Rightarrow 0 < a^n < \frac{1}{nh}$$

By squeeze play

$$\lim_{n \rightarrow \infty} a^n = 0$$

Case IV

if $a = 0$

$$\lim_{n \rightarrow \infty} a^n = 0 \Rightarrow \{a^n\} \text{ Converges to } 0$$

Case V

Let $-1 < a < 0$

put $a = -b$, then

$$-1 < a < 0 \Rightarrow -1 < -b < 0$$

$$\Rightarrow 0 < b < 1$$

$$b^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} a^n = \lim_{n \rightarrow \infty} (-b)^n = \lim_{n \rightarrow \infty} (-1)^n b^n = 0$$

$\Rightarrow \{a^n\}$ Converges to 0

Case VI

Let $a = -1$

$$a^n = (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

The sequence is $-1, 1,$

$\Rightarrow \{a^n\}$ oscillates finitely

Case VII $a \leq -1$ let $a = -b$
 ~~$a = -1$~~

$$a < -1 \Rightarrow -b < -1 \Rightarrow b > 1$$

By $b^n \rightarrow \infty$ as $n \rightarrow \infty$

$$a^n = (-b)^n = \begin{cases} -b^n & \text{if } n \text{ is odd} \\ b^n & \text{if } n \text{ is even} \end{cases}$$

$$a^n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ when } n \text{ is odd}$$

$$a^n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ when } n \text{ is even}$$

$\Rightarrow \{a^n\}$ oscillates infinitely

Hence $\{a^n\}$ converges when $-1 < a \leq 1$

Lemma # Let a & b be numbers such that $0 \leq a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n$$

OR $b^n \times \{b - (n+1)(b-a)\} < a^{n+1}$

Proof $0 \leq a < b$

By actual division

$$\frac{b^{n+1} - a^{n+1}}{b - a} = a^0 b^n + a b^{n-1} + a^2 b^{n-2} + \dots + a^n b^0 \quad (n \text{ terms})$$

$$< b^n + b b^{n-1} + b^2 b^{n-2} + b^3 b^{n-3} + \dots + b^{n-1} b + b^n$$

$$= (n+1)b^n$$

$$\Rightarrow b^{n+1} - a^{n+1} < (n+1)b^n [b - a]$$

$$b^{n+1} - (n+1)b^n(b-a) \stackrel{148}{<} a^{n+1}$$

$$b^n [b - (b-a)(n+1)] < a^{n+1}$$

Theorem # prove that the sequence $\{(1+\frac{1}{n})^n\}$ is bounded and increasing

OR
Let $e_n = (1+\frac{1}{n})^n$, prove that $\{e_n\}$ is increasing and bounded.

Ca. Proof # we know that for $0 \leq a < b$
Taking $b^n [b - (n+1)(b-a)] < a^{n+1}$ $\neq b = 1 + \frac{1}{n}$
 $a = 1 + \frac{1}{n+1}$
 $b > a \quad \therefore \{\frac{1}{n+1} < \frac{1}{n}\}$

$$(1+\frac{1}{n})^n [(1+\frac{1}{n}) - (n+1)(1+\frac{1}{n} - 1 - \frac{1}{n+1})] < (1+\frac{1}{n+1})^{n+1}$$

$$(1+\frac{1}{n})^n [(1+\frac{1}{n}) - (n+1)(\frac{1}{n} - \frac{1}{n+1})] < (1+\frac{1}{n+1})^{n+1}$$

$$(1+\frac{1}{n})^n [(1+\frac{1}{n}) - (n+1)(\frac{n+1-n}{n(n+1)})] < (1+\frac{1}{n+1})^{n+1}$$

$$(1+\frac{1}{n})^n [1+\frac{1}{n} - \frac{1}{n}] < (1+\frac{1}{n+1})^{n+1}$$

$$(1+\frac{1}{n})^n < (1+\frac{1}{n+1})^{n+1}$$

$$\Rightarrow e_n < e_{n+1} \quad \forall n$$

$$e_n =$$

$\Rightarrow \{e_n\}$ is \nearrow 149

Let $a = 1$

$$b = 1 + \frac{1}{2n}$$

$$\left(1 + \frac{1}{2n}\right)^n \left\{ 1 + \frac{1}{2n} - (n+1) \left(1 + \frac{1}{2n} - 1\right) \right\} < 1^{n+1}$$

$$\left(1 + \frac{1}{2n}\right)^n \left[1 + \frac{1}{2n} - \frac{1}{2} - \frac{1}{2n} \right] < 1$$

$$\left(1 + \frac{1}{2n}\right)^n \left[\frac{1}{2} \right] < 1$$

29 $\Rightarrow \left(1 + \frac{1}{2n}\right)^n < 2$

$\Rightarrow \left(1 + \frac{1}{2n}\right)^{2n} < 4 \Rightarrow e_{2n} < 4$

This not Adjacent
Remedize.

Since $\{e_n\}$ is increasing and $2n > n$,
therefore

$$e_n < e_{2n} < 4 \quad \forall n$$

$\Rightarrow \{e_n\}$ is bounded.

Thus $\{e_n\}$ is convergent.

24

If $n = 1$ $e_1 = (1+1)^1 = 2$.

$$\Rightarrow 2 \leq e_n < 4 \quad \forall n$$

$\therefore n$ is the OR integer, by binomial theorem.

$$e_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \dots + \frac{n(n-1)\dots 2 \cdot 1}{n!} \cdot \frac{1}{n^n}$$

$$e_n = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots$$

$$+ \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right)$$

changing n to $n+1$

$$e_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) + \frac{1}{3!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right)$$

$$+ \dots + \frac{1}{n!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \dots \left(1 - \frac{n-1}{n+1}\right)$$

$$+ \frac{1}{(n+1)!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \dots \left(1 - \frac{n}{n+1}\right)$$

note that e_n has $n+1$ terms & e_{n+1} has $n+2$ terms and one more term.

$$\text{Also } \frac{1}{n+1} > \frac{1}{n} \Rightarrow -\frac{1}{n+1} > -\frac{1}{n}$$

$$\Rightarrow 1 - \frac{1}{n+1} > 1 - \frac{1}{n}$$

$$\text{Similarly } 1 - \frac{k}{n+1} > 1 - \frac{k}{n} \text{ for } k=1, 2, \dots, n-1$$

\Rightarrow each term in e_n after the 1st two terms is less than the corresponding term in e_{n+1} and e_{n+1} has also one additional term. It comes out

$$e_{n+1} > e_n$$

$$\Rightarrow \{e_n\} \text{ is } \uparrow$$

To show that $\{e_n\}$ is bounded above we have

$$e_n = 1 + n \cdot \frac{1}{n} + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots$$

$$+ \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right)$$

$$\leq 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} + \dots + \frac{1}{n!} \quad \because 1 - p_k < 1$$

$$\leq 1 + \frac{1}{2^0} + \frac{1}{2^1} + \dots + \frac{1}{2^{k-1}} + \dots + \frac{1}{2^{n-1}} \quad \because \frac{1}{p!} \leq \frac{1}{2^{p-1}}$$

$$\Rightarrow e_n$$

$$= 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 1 + 2[1 - (\frac{1}{2})^n] < 3$$

$$\Rightarrow e_n < 3 \quad \forall n$$

$$\text{for } n=1$$

$$e_1 = 2$$

$$\text{for } n > 1$$

$$2 < e_n < 3 \quad \forall n > 1$$

Thus $\{e_n\}$ bounded monotone and hence convergent

Note By refining our estimates we can find closer rational approximations to e but we can not evaluate e since e is an irrational number. However it is possible to evaluate e to as many decimal places as desired.

Theorem # Prove that e is irrational

Proof # Let e be a rational number.

$$\text{and } e = \frac{p}{q} \quad \text{where } p, q \in \mathbb{N} \text{ and } q \geq 1$$

$$\text{Let } s_q = \sum_{k=1}^q \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{q!}$$

$$\text{Since } e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$\text{So } e - s_q = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \dots$$

$$= \frac{1}{(q+1)!} \left[1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \dots \right]$$

$$= \frac{1}{(q+1)!} \left[1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \dots \right]$$

$$= 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 1 + 2[1 - (\frac{1}{2})^n] < 3$$

$$\Rightarrow e_n < 3 \quad \forall n$$

$$\text{for } n=1 \quad e_1 = 2$$

$$\text{for } n > 1$$

$$2 < e_n < 3 \quad \forall n > 1$$

Thus $\{e_n\}$ bounded monotone and hence convergent

Note By refining our estimates we can find closer rational approximations to e but we can not evaluate e since e is an irrational number. However it is possible to evaluate e to as many decimal places as desired.

Theorem # Prove that e is irrational

Proof # Let e be a rational number.

$$\text{and } e = \frac{p}{q} \quad \text{where } p, q \in \mathbb{N} \text{ and } q \geq 1$$

$$\text{Let } s_q = \sum_{k=1}^q \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{q!}$$

$$\text{Since } e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$\text{So } e - s_q = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \dots$$

$$= \frac{1}{(q+1)!} \left[1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \dots \right]$$

$$< \frac{1}{(q+1)!} \left[1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \dots \right]$$

$$a_n \leq a_m < -k \quad \forall n \geq m$$

$$\Rightarrow a_n < -k \quad \forall n \geq m$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = -\infty \quad \Rightarrow \{a_n\} \text{ diverges to } -\infty$$

$$= \frac{1}{(2+1)!} \left[\frac{1}{1 - \frac{1}{2+1}} \right] \quad \underline{152}$$

$$= \frac{1}{(2+1)!} \times \frac{2+1}{2} = \frac{1}{(2!)2}$$

$$s_0 = \frac{9}{1-2}$$

$$\Rightarrow 0 < (e - s_2) < \frac{1}{2!2}$$

$$\Rightarrow 0 < (e - s_2) 2! < \frac{1}{2} < 1 \quad \left(\because \frac{2!}{2} < 1 \right)$$

$$0 < e 2! - s_2(2!) < 1$$

$$\therefore 2e = p \quad \text{by } p_2 = e$$

$$\Rightarrow e 2! = e 2(2-1)! = p(2-1)! \text{ is an integer}$$

Also

$$(2!) s_2 = 2! \left[1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{2!} \right]$$

$$= 2! + 2! + \frac{2!}{2!} + \frac{2!}{3!} + \dots + 1 \text{ is}$$

an integer for each $2 > 1$

$\Rightarrow (e - s_2) 2!$ is an integer lying b/w 0 and 1 which is a contradiction because there is no integer bet 0 & 1.

Hence $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ is an ~~int~~ irrational no.

$$\Rightarrow \quad \quad \quad 2^{2n-1} e - \dots + \frac{1}{2^{2n-1}} \quad \therefore \frac{1}{p!} \leq \frac{1}{2^{p-1}}$$

153

Theorem #1 (i) Every monotonically increasing sequence which is not bounded above diverges to $+\infty$ i.e. diverges properly.

(ii) Every monotonically decreasing sequence which is not bounded below diverges to $-\infty$.

Proof #1 (i) Let $\{a_n\}$ is \uparrow which is not bounded above. Then, given any $K > 0$, however large, \exists a true integer m such that

$$a_m > K.$$

$$\therefore \{a_n\} \text{ is } \uparrow$$

$$a_n \geq a_m > K \quad \forall n \geq m.$$

$$\Rightarrow a_n > K \quad \forall n \geq m.$$

Thus for every real no $K > 0$, however large, we have a true integer m s.t.

$$a_n > K \quad \forall n \geq m.$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = \infty$$

$$\Rightarrow \{a_n\} \text{ diverges to } +\infty$$

(ii) Let $\{a_n\} \downarrow$ which is not bounded below. Then given any $K > 0$, large, \exists a true integer m s.t.

$$a_m < -K$$

$$\therefore \{a_n\} \downarrow$$

$$a_n \leq a_m < -K \quad \forall n \geq m.$$

$$\Rightarrow a_n < -K \quad \forall n \geq m.$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = -\infty \Rightarrow \{a_n\} \text{ diverges to } -\infty$$

Theorem Every monotone ¹⁵⁴ sequence either converges or diverges OR A monotone sequence is never oscillatory.

Proof let $\{a_n\}$ be a monotone sequence, then either $\{a_n\} \uparrow$ or $\{a_n\} \downarrow$

Case I let $\{a_n\} \uparrow$

If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to l.u.b. If $\{a_n\}$ is not bounded, then it diverges to $+\infty$

Case II let $\{a_n\} \downarrow$

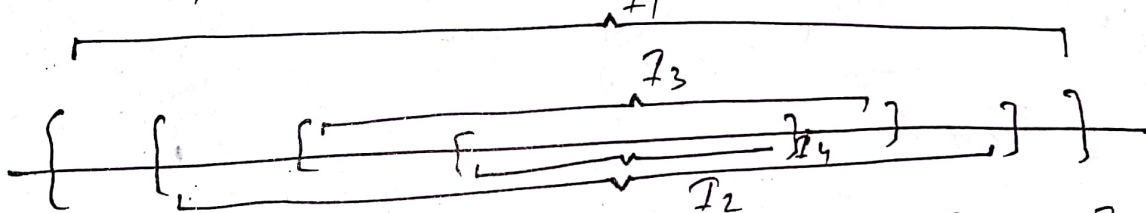
If $\{a_n\}$ is bounded below, then $\{a_n\}$ is cgt to g.l.b. If $\{a_n\}$ is not bounded below, then $\{a_n\}$ is dgt to $-\infty$

Nested Intervals

A sequence $\{I_n = [a_n, b_n]\}$ of closed intervals is called nested if

$$I_{n+1} \subseteq I_n \text{ i.e.}$$

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \dots$$



e.g. $I_n = [0, 1/n]$ $\forall n \in \mathbb{N}$, then $I_n \supseteq I_{n+1}$

$\forall n$
In this case 0 belongs to all I_n and 0 is the only such common point i.e. $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$

$$\begin{aligned} & + \frac{1}{2^{k+1}} \leq \dots + \frac{1}{2^n} < 1 \\ & \leq 1 \end{aligned} \quad \therefore \frac{1}{p_1} \leq \frac{1}{2^{p-1}}$$

In general, a ¹⁵⁵ nested interval sequence need not have common point e.g.

If $I_n = (0, 1/n)$ $n \in \mathbb{N}$, this sequence $\{I_n\}$ is nested but there is no common point because for every $x > 0$ $\exists m \in \mathbb{N}$ such that

$$1/m < x \quad (\text{Archimedean})$$

so that $x \notin I_n$. Similarly the sequence of intervals $K_n = (n, \infty)$ $n \in \mathbb{N}$ is nested and have no common point.

However, it is an important property of \mathbb{R} that every ~~closed~~ sequence of closed, bounded intervals does have a common point

Nested Interval Property

(Cantor's Nested interval theorem)

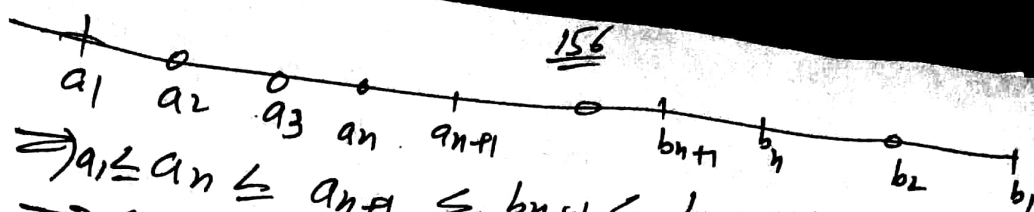
Let $\{I_n = [a_n, b_n]\}$ be a sequence of closed intervals such that

- (a) $I_{n+1} \subseteq I_n$ i.e. sequence of nested intervals
- (b) $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ contains

unique point i.e. there is exactly ^{$n=1$} one real no. common to all intervals I_n .

Proof # Since $I_{n+1} \subseteq I_n$

$$\Rightarrow [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad \forall n \in \mathbb{N}$$



$\Rightarrow a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1, \forall n \in \mathbb{N}$

$\Rightarrow \{a_n\}$ is bounded monotone increasing and $\{b_n\}$ is bounded monotone decreasing sequence.

Thus $\{a_n\}$ & $\{b_n\}$ Converge.

Let $\lim_{n \rightarrow \infty} a_n = a$ (Lub) $\lim_{n \rightarrow \infty} b_n = b$ (glb)

$$\therefore b_n = (b_n - a_n) + a_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (b_n - a_n) + \lim_{n \rightarrow \infty} a_n$$

$$b = 0 + a \quad \left[\because \lim_{n \rightarrow \infty} (b_n - a_n) = 0 \right]$$

$$b = a = x \text{ (say)}$$

Now x is Lub of $\{a_n\}$ &

$$a_n \leq x \quad \forall n$$

Also x is glb of $\{b_n\}$

$$x \leq b_n \quad \forall n.$$

$$\Rightarrow a_n \leq x \leq b_n \quad \forall n.$$

$$\Rightarrow x \in [a_n, b_n] \quad \forall n$$

$$\Rightarrow x \in I_n \quad \forall n.$$

$$\Rightarrow x \in \bigcap_{n=1}^{\infty} I_n$$

$\Rightarrow \exists$ a number common to all intervals.

Uniqueness If possible let x, y be two distinct numbers common to all the intervals.

Then $x \in [a_n, b_n] \quad \forall n$ $y \in [a_n, b_n] \quad \forall n$
 If $x < y$, then $a_n \leq x < y \leq b_n \quad \forall n$

$$+ \frac{1}{2^{k+1}} - \dots - + \frac{1}{2^{n-1}} \quad \therefore \frac{1}{p!} \leq \frac{1}{2^{p-1}}$$

$$\Rightarrow \subseteq \eta -$$

$$\Rightarrow y - x \leq b_n - a_n \quad \forall n.$$

Let $\epsilon = y - x > 0$, then

$$b_n - a_n \geq \epsilon \quad \forall n. \rightarrow \textcircled{1}$$

$$\text{Also } \lim_{n \rightarrow \infty} (b_n - a_n) = 0$$

\Rightarrow for $\epsilon = y - x$ there must exist an integer m such that

$$|(b_n - a_n) - 0| < \epsilon \quad \forall n \geq m.$$

$$\Rightarrow b_n - a_n < \epsilon \quad \forall n \geq m$$

which contradicts $\textcircled{1}$. Hence x is only element common to all intervals.

Note # The word closed in above theorem can not be dropped i.e. the intersection of a decreasing sequence of open intervals may be empty.

e.g. $I_n = (0, 1/n) \quad \forall n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = \phi$

Nested Interval Property

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, then \exists a no $x_0 \in \mathbb{R}$ such that $x_0 \in I_n \quad \forall n \in \mathbb{N}$.

Proof \because Intervals are nested, we have

$$I_n \subseteq I_1 \quad \forall n \in \mathbb{N}.$$

So that $a_n \leq b_1 \quad \forall n$ & $a_1 \leq b_n \quad \forall n$

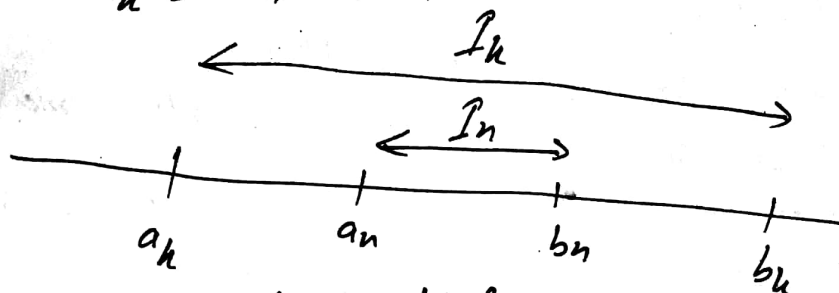
Hence the non-empty set $\{a_n : n \in \mathbb{N}\}$ is bounded above and let x_0 be l.u.b of this set

$$\text{Then } a_n \leq x_0 \leq b_n \quad \forall n$$

, finitely many peak points (peaks) or valleys

we claim that 158
 Because for any partial $x_0 \leq b_n$ ^{we show that}
 upper bound of $\{a_k: k \in \mathbb{N}\}$.
 If $n \leq k$, then $I_n \supseteq I_k$, we have.

$a_k \leq b_k \leq b_n$
 If $k < n$, then since $I_k \supseteq I_n$, we have.
 $a_k \leq a_n \leq b_n$



Thus $a_k \leq b_n \quad \forall k$

$\Rightarrow b_n$ is an upper bound of the set $\{a_k: k \in \mathbb{N}\}$

Hence $x_0 \leq b_n$ for each $n \in \mathbb{N}$

$\Rightarrow a_n \leq x_0 \leq b_n \quad \forall n$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} I_n$

Note x_0 above may not be unique.

Peak Point and Peak of a sequence.

A natural no m is called a peak point of the sequence $\{a_n\}$ if

$$a_n \leq a_m \quad \forall n \geq m$$

and the term (a_m) is called a peak of sequence. i.e. a_m is never exceeded by any term that follows it in the sequence

\Rightarrow $a_n \leq a_m$

Note (a) in a decreasing sequence every term is a peak and every natural no is a peak point.

(b) In an increasing sequence no term is peak and no natural no is a peak point

e.g. $a_n = \frac{1}{n}$ when $n \leq 5$

Then 1, 2, 3, 4, 5 are five peak points

$$a_n = -n \quad \text{when } n > 5$$

1, 2, 3, 4, 5 are five peak points

(i) If $a_n = 1$ when $n = 1, 2, \dots, m$
 $= -1$ when $n > m$

Then m is only peak point.

(iii) If $a_n = \frac{1}{n}$, then every natural no is a peak point

because for any natural no m , then for $n > m$

$$\frac{1}{n} < \frac{1}{m} \quad \text{i.e. } a_n < a_m \quad \forall n > m$$

Thus a sequence may have no peak point, a finite no of peak points or an infinite no of peak points.

Monotone Subsequence Theorem

Theorem # Every sequence of real nos contain a monotone subsequence.

Proof # Let $\{a_n\}$ be any sequence.
 Sequence $\{a_n\}$ may have no peak point (peak)
) finitely many peak points (peaks) or infinitely many

points.

160

The sequence has no peak point and no peak.

1 is not a peak point, \exists a natural $n_1 > 1$ such that $a_{n_1} > a_1$.
 n_1 is not a peak point, \exists a natural $n_2 > n_1$ s.t. that $a_{n_2} > a_{n_1}$.
Repeating the above argument, we get a sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \dots \quad n_1 = 1$$

the sequence $\{a_n\}$ contains a monotone increasing subsequence.

The sequence $\{a_n\}$ has a finite no. of peak points.

Let m be the largest peak point.

a_m is a peak. Let n_1 be a natural no. s.t. $n_1 > m$, then n_1 is not a peak point.

\exists a natural no. $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$.

n_2 is not a peak point.

\exists a natural no. $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$.

Repeating the above argument we get a sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \dots$$

the sequence $\{a_n\}$ contains a monotone increasing subsequence.

The sequence $\{a_n\}$ has an infinite no. of peak points.

A term (a_m) is called a peak if a_m is never exceeded by any term that follows it in the sequence.

peak points.

160

Case I The sequence has no peak point and hence no peak.

Since 1 is not a peak point, \exists a natural no $n_2 > 1$ such that $a_{n_2} > a_1$

$\therefore n_2$ is not a peak point, \exists a natural no $n_3 > n_2$ s.t. that $a_{n_3} > a_{n_2}$

Repeating the above argument, we get a sub-sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \dots \dots \dots n_1 = 1$$

Thus the sequence $\{a_n\}$ contains a monotonically increasing subsequence.

Case II # The sequence $\{a_n\}$ has a finite no of peak points.

Let m be the largest peak point and a_m is a peak. Let n_1 be a natural no s.t. that $n_1 > m$, then n_1 is not a peak point

$\therefore \exists$ a natural no $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$

Again n_2 is not a peak point

$\therefore \exists$ a natural no $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$

Repeating the above argument we get a sub-sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \dots \dots \dots$$

Thus $\{a_n\}$ contains a monotone increasing sub-sequence.

Case III The sequence $\{a_n\}$ has an infinite no of peak points.

and the term (a_m) is called a peak point. i.e. a_m is never exceeded by any term that follows it in the sequence

167

Let the peak points be n_1, n_2, n_3, \dots
 such that $n_1 < n_2 < n_3 < \dots$

$\therefore n_1$ is ~~not~~ a peak point and $n_2 > n_1$

$$\therefore a_{n_2} < a_{n_1}$$

$\therefore n_2$ is not a peak point & $n_3 > n_2$

$$\therefore a_{n_3} < a_{n_2}$$

Repeating the above process we get a
 sub-sequence $\{a_{n_k}\}$ so that $a_{n_1} > a_{n_2} > a_{n_3} > \dots$

Thus the sequence $\{a_n\}$ contains a monotonically
 decreasing sequence $\{a_{n_k}\}$

Theorem (Bolzano Weierstrass)

Every bounded real sequence has a convergent
 subsequence

Proof # Let $\{a_n\}$ be bounded sequence.

Then there is a closed interval $I_0 = [a, b]$
 such that $a_n \in [a, b] = I_0 \quad \forall n$

Bisecting $I_0 = [a, b]$ into two equal intervals

$$[a, \frac{a+b}{2}], [\frac{a+b}{2}, b]$$

one of these intervals must contain a_n for infinite
 many n . Let this interval be I_1

$$I_1 = [a, \frac{a+b}{2}] = [a, b] \text{ with length } b-a = \frac{b-a}{2}$$

$$\text{and } I_0 \supset I_1$$

Bisecting $I_1 = [a_1, b_1]$ into two equal intervals

one of these contains infinite terms of the sequence $\{a_n\}$. Let this be

$$I_2 = [a_1, \frac{a_1+b_1}{2}] \quad \text{with length} \quad \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$$

and $I_1 \supset I_2$

Continuing this process, we obtain a sequence of nested intervals I_0, I_1, I_2, \dots such that

$$I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots$$

with $b_k - a_k = \frac{b-a}{2^k} \rightarrow 0$ as $n \rightarrow \infty$

and each interval contains an infinite number of terms of the sequence.

Choose a true integer n_1 such that

$$a_{n_1} \in I_1$$

$\therefore I_2 = [a_2, b_2]$ contains infinite terms of $\{a_n\}$

$\therefore \exists$ true integer $n_2 > n_1$ such that

$$a_{n_2} \in I_2$$

Continuing this process we obtain numbers

$a_{n_1}, a_{n_2}, a_{n_3}, \dots$ such that

$$a_{n_k} \in [a_k, b_k] \quad k=1, 2, 3, \dots$$

and $n_1 < n_2 < n_3 < n_4 < \dots$

of peak points

and the term (a_{n_k}) is called a peak point. i.e. a_{n_k} is never exceeded by any term that follows it in the sequence.

Proof $\therefore \{a_n\}$ is ¹⁶⁴ bounded
 \therefore By B.W Theorem There is
 a convergent sub-sequence $\{a_{n_k}\}$.

Also $a_{n_k} > \alpha \quad \forall n_k > n_0$

Hence $\lim_{k \rightarrow \infty} a_{n_k} \geq \alpha$

Cantor's Intersection Theorem.

for real line.

Theorem # If $F = \{F_n\}$ is a countable
 class/family of non-empty closed and bounded
 sets such that

$$F_1 \supset F_2 \supset F_3 \supset F_4 \dots \supset F_n \supset \dots$$

Then $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

Proof # \therefore each F_n is a non-empty
 closed and bounded set

$\therefore \exists$ exists sequences of real
 nos M_n & m_n belong to F_n
 such that

$$M_n = \sup F_n \quad m_n = \inf F_n$$

$$\therefore F_n \supset F_{n+1} \quad \forall n$$

$$\therefore M_n \geq M_{n+1} \quad \& \quad m_n \leq m_{n+1} \quad \forall n \in \mathbb{N}$$

Thus $\{a_{n_k}\}$ is a ¹⁶³subsequence of a sequence $\{a_n\}$

we show that $\{a_{n_k}\}$ is cgt.

The sequence $\{a_n\}$ is monotone and bounded and so is cgt.

$$\text{Let } l = \lim_{k \rightarrow \infty} c_k.$$

Similarly the sequence $\{d_k\}$ converges to some no m

$$m - l = \lim_{k \rightarrow \infty} d_k - \lim_{k \rightarrow \infty} c_k.$$

$$= \lim_{k \rightarrow \infty} (d_k - c_k) = \lim_{k \rightarrow \infty} \frac{d - c}{2^k} = 0$$

$$m = l$$

$$\therefore a_{n_k} \in [c_k, d_k] \quad k=1, 2, 3, \dots$$

$$c_k \leq a_{n_k} \leq d_k.$$

By squeeze theorem

$$\lim_{k \rightarrow \infty} a_{n_k} = l = m$$

Thus $\{a_n\}$ contains a convergent subsequence

Corollary Suppose that $\{a_n\}$ is bounded sequence

such that $a_n > d \quad \forall n > n_0$

where d is a real number, then there is a convergent subsequence $\{a_{n_k}\}$ such that

$$d \leq \lim_{k \rightarrow \infty} a_{n_k}$$

ence that follows it in the sequence

165

Now the lowest bound for $\bigcap_{n=1}^{\infty} F_n$ is the lower bound of the sequence $\{M_n\}$ of upper bounds. Thus $\{M_n\}$ is non-increasing sequence which is bounded below and is therefore convergent.

$$\text{Let } \lim_{n \rightarrow \infty} M_n = M.$$

We show that $M \in \bigcap_{n=1}^{\infty} F_n$

$$\text{Let } M \notin \bigcap_{n=1}^{\infty} F_n$$

Then there will be at least one neighborhood say $]M - \epsilon, M + \epsilon[$ $\epsilon > 0$ which contains no point of $\bigcap_{n=1}^{\infty} F_n$

$\Rightarrow]M - \epsilon, M + \epsilon[$ contains no point of F_n for some value of n say m

$\Rightarrow]M - \epsilon, M + \epsilon[$ contains no point of F_n for $n \geq m$

$$\Rightarrow M_n \notin]M - \epsilon, M + \epsilon[\quad \forall n \geq m$$

Contradicting the fact that $\{M_n\}$ converges to M

$$\text{Hence } M \in \bigcap_{n=1}^{\infty} F_n$$

Cluster points (Limit points) of a sequence

Frequently valid property # A property of statement $P(n)$ is frequently valid.

For a sequence $\{a_n\}$ ¹⁶⁶ if for every natural no $m \exists$ at least one $m, > m$ such that $P(n)$ is true.

Note An eventually valid statement is also frequently valid.

Cluster point

A real no c is said to be a cluster point of a sequence $\{a_n\}$ if every nbd \exists c contains infinitely many terms of the sequence. i.e.

$$\forall \epsilon > 0 \quad a_n \in (c - \epsilon, c + \epsilon) \text{ for infinitely many values of } n$$

Note A cluster point of a sequence is called a limit point or a condensation point or an accumulation point or a subsequential limit of the sequence.

Difference b/w limit and limit point of a sequence

If $l \in \mathbb{R}$ is the limit of a sequence $\{a_n\}$, then for $\epsilon > 0 \exists m \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon \quad \forall n > m$$

\Rightarrow every nbd of l contains all except a finite no of terms of the sequence. i.e. There are

$$\alpha \leq \lim_{k \rightarrow \infty} a_k$$

that follows is in the ...

only a finite no of terms outside each nbd of l .
 whereas if l is a limit point of sequence $\{a_n\}$, then every nbd of l contains infinitely many terms of the sequence and there may be infinite terms out the nbd i.e it does not exclude the possibility of an infinite no of terms of sequence lying outside the interval or nbd

Hence limit of a sequence is a limit of the sequence but a limit point of a sequence need not be the limit of the sequence. e.g

$\{(-1)^n\}$ has limit points or cluster point $1, -1$ but has no limit

Note # (1) If $a_n = l$ for infinitely many values of n , then l is a limit point of $\{a_n\}$

(2) If for an $\epsilon > 0$, $(l - \epsilon, l + \epsilon)$ for finitely many values of n , then l can not be a cluster point of $\{a_n\}$

(3) Limit point of a sequence need not be a sequence.

Def 2 A real no l is called a cluster point of a sequence $\{a_n\}$ if given $\epsilon > 0$ and a true integer $m \exists$ a no $k > m$ s.t

$$|a_k - l| < \epsilon$$

Thus every nbd of l contains a term of the sequence. This is equivalent to saying that

every nbd of l contains infinitely many terms of the sequence because if $(l-\epsilon, l+\epsilon)$ contains only finite no of terms of $\{a_n\}$ say a_1, a_2, \dots, a_n

Let $\delta = \min\{|l-a_1|, |l-a_2|, \dots, |l-a_n|\}$
 now $(l-\delta, l+\delta)$ contains no term of the sequence which is a contradiction.

clearly above two definitions are equivalent

Def 3 A real no l is called a cluster point of a sequence $\{a_n\}$ if l is limit of some subsequence of $\{a_n\}$.

Note cluster point is also called a subsequential limit

Limit point of range set & cluster point

We note that limit point of the range set $S = \{a_1, a_2, a_3, \dots\}$ is automatically a cluster point of $\{a_n\}$. The two notions differ only in that for limit points the nbd is deleted where as for cluster point it is not. The distinction is also introduced to cover the possibility that terms of a sequence may be repeated frequently and the range set may be finite and has no limit point whereas sequence has infinite terms and may have cluster point e.g. for sequence $(-1)^n$ the range set $\{-1, 1\}$ has no limit point but $-1, 1$ are cluster point. Thus if a sequence

term that follows in ...

has a limit point l ¹⁶⁹ then l is a cluster point but the converse may not be usually true e.g. $\{(-1)^n\}$ has $+1, -1$ cluster points but has no limit point.

Example 1 0 is a limit point of the sequence

Sol $\{\frac{1}{n}\}$

For $\epsilon > 0$ $\exists m \in \mathbb{N}$ s.t. that $\frac{1}{m} < \epsilon$

\therefore for $n \geq m$ $0 < \frac{1}{n} \leq \frac{1}{m} < \epsilon$

$\Rightarrow -\epsilon < 0 < \frac{1}{n} < \epsilon \quad \forall n \geq m$

$\Rightarrow \frac{1}{n} \in (-\epsilon, \epsilon) \quad \forall n \geq m$

\Rightarrow Every nbd of 0 contains infinitely many terms of the sequence $\{\frac{1}{n}\}$

Example 2 $\#$ The sequence $\{(-1)^n\}$ has two limit points

Example 3 $\#$ The sequence $\{n\}$ has no cluster point

Theorem $\#$ If l is a limit point of the range of a sequence $\{a_n\}$, then l is a limit point of the sequence $\{a_n\}$

Proof $\#$ Let $S = \text{range of } \{a_n\} = \{a_n : n \in \mathbb{N}\}$

$\therefore l$ is a limit point of S

\therefore Every deleted nbd of S contains infinitely many elements of S which are terms of $\{a_n\}$

\Rightarrow Every nbd of l contains infinitely many terms of $\{a_n\}$

$\Rightarrow l$ is a limit point of the sequence $\{a_n\}$

Note $\#$ Converse of the above theorem may not be true

Consider

170

$$a_n = 1 + (-1)^n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even} \end{cases}$$

0, 2 are the limit points of the sequence.

But the range $= \{0, 2\}$ is a finite set and finite set has no limit point

(2) If the terms of the sequence are distinct, then the limit points of the sequence are the limit points of the range set.

Theorem # If a sequence converges to l , then l is the only limit point of the sequence

Proof # The sequence $\{a_n\}$ converges to l

\Rightarrow Given $\epsilon > 0$ \exists an integer m such that

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

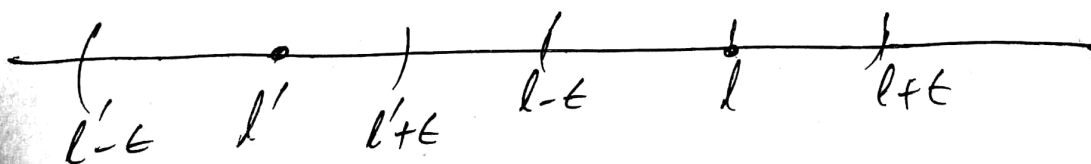
$$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m \quad (1)$$

$\Rightarrow a_n \in (l - \epsilon, l + \epsilon)$ for infinitely many values of n .

\Rightarrow every nbhd of l contains infinitely many terms of the sequence $\{a_n\}$

$\Rightarrow l$ is a limit point of the sequence $\{a_n\}$

If possible let l' be another limit point of the sequence $\{a_n\}$



but

term that follows

Let $\epsilon = \frac{1}{3}(l - l')$ where $l > l'$
 Then $(l' - \epsilon, l' + \epsilon) \cap (l - \epsilon, l + \epsilon) = \emptyset$
 From ① $a_n \in (l - \epsilon, l + \epsilon) \quad \forall n \geq m$

$\therefore a_n \in (l' - \epsilon, l' + \epsilon)$ for almost all values of n

\Rightarrow finitely many terms of $\{a_n\}$ lie in $(l' - \epsilon, l' + \epsilon)$

Hence l' is not a limit point of the sequence

Hence l is only limit point of the sequence.

Theorem (Bolzano Weierstrass theorem)

Every bounded sequence has at least one limit point

Proof # Let $\{a_n\}$ be a bounded sequence.

and S its range i.e.

$$S = \{a_n : n \in \mathbb{N}\}$$

$\therefore \{a_n\}$ is bounded.

$\therefore S$ is bounded.

Case I # Let S be a finite set

Then \exists a real no l such that $a_n = l$ for any infinite no of values of $n \in \mathbb{N}$.

\Rightarrow Given $\epsilon > 0$, $a_n \in (l - \epsilon, l + \epsilon)$ for an infinite no of values of n

\Rightarrow Every nbd of l contains infinitely many terms of the sequence $\{a_n\}$

$\Rightarrow l$ is a limit point of the sequence $\{a_n\}$

\Rightarrow l is a limit point of the sequence $\{a_n\}$

Case II

Let S be ¹⁷² an infinite set
Since S is an infinite bounded set, by
B.W. theorem for sets S has at least one
limit point say l

Now l is a limit point of S
 \Rightarrow Every nbd of l contains an infinite no
of elements of S

But each term of S is a term of $\{a_n\}$
 \therefore Every nbd of l contains an infinite
no of terms of the sequence $\{a_n\}$

$\Rightarrow l$ is a limit point of the sequence $\{a_n\}$

Corollary # If S is a closed and bounded
(i.e. compact) set, then every sequence in S
has a limit point

Proof # A sequence $\{a_n\}$ in S if $a_n \in S \forall n$

Let $\{a_n\}$ be a sequence in S , then $a_n \in S \forall n$
Since S is bounded, the sequence $\{a_n\}$ is
bounded and consequently it has a limit
point say l by B.W. theorem.

We show that $l \in S$

Let $l \in S^c$, then S being closed, S^c is open

$\therefore S^c$ is nbd of l

But S^c contains no term of $\{a_n\}$. This contradicts
the fact that l is a limit point of $\{a_n\}$

$\therefore l \notin S^c$. Hence $l \in S$

term that follows

494 Stratf
WARNING: TO AVOID
Corollary 2 # of 173 If I is a closed interval, then any sequence in I has a limit point in I .

Proof # $\because I$ is closed interval

$\therefore I$ is closed and bounded.

\Rightarrow The result follows from Corollary 1.

Theorem # The set of limit points of a bounded sequence is bounded.

Proof let $\{a_n\}$ be a bounded sequence

$\Rightarrow \exists$ real no $k \neq K$ ($k \leq K$)

such that $k \leq a_n \leq K \quad \forall n \in \mathbb{N}$

$\therefore a_n \notin (-\infty, k) \text{ \& } a_n \notin (K, \infty)$

for any n

Let l be any real no

If $l \in (-\infty, k)$, then $(-\infty, k)$ contains no terms of the sequence & l is not limit point of $\{a_n\}$

If $l \in (K, \infty)$, then (K, ∞) contains no term of sequence $\{a_n\}$ and l is not limit point of $\{a_n\}$

Thus no point outside $[k, K]$ is a limit point of $\{a_n\}$

\Rightarrow The limit points of $\{a_n\}$ lie in $[k, K]$

\Rightarrow The set of all the limit points of a bounded sequence is bounded.

Note The bounds of the set of limit points of a bounded sequence are same as the bounds of the sequence.

$\therefore \inf \{a_n\} \leq \liminf a_n \leq \limsup a_n \leq \sup \{a_n\}$

(2) The set of ¹⁷⁴limit points of an unbounded sequence may or may not be bounded.
 e.g. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is unbounded but the set of limit points $\{0\}$ is bounded.
 The sequence $\{2, 1+\frac{1}{2}, 2+\frac{1}{2}, 2+\frac{1}{3}, \dots\}$ is unbounded and the set of limit points is \mathbb{N} which is unbounded.

Theorem # Every bounded sequence has the greatest and the least limit points.

Proof # Let $\{a_n\}$ be a bounded sequence.

Then ~~$\{a_n\}$~~ the set E of limit points is also bounded and $E \neq \emptyset$ (B.W. Theorem)

By Completeness property, E has minimum and

Supremum.

Let $\inf E = l$ & $\sup E = u$

For $\epsilon > 0$ let $(u-\epsilon, u+\epsilon)$ be nbd of u .

$\sup E = u \Rightarrow \exists$ some $x \in E$ s.t. that

$$u-\epsilon < x \leq u < u+\epsilon$$

$$\Rightarrow x \in (u-\epsilon, u+\epsilon)$$

$\Rightarrow (u-\epsilon, u+\epsilon)$ is a nbd of x

$\therefore x \in E$ is a limit point of $\{a_n\}$

\Rightarrow every nbd of x contains infinitely many terms of

$\{a_n\} \Rightarrow (u-\epsilon, u+\epsilon)$ contains infinite terms of $\{a_n\}$

This is true for every $\epsilon > 0$

$\Rightarrow u$ is a limit point of $\{a_n\}$

$\Rightarrow u \in E$

Similarly $l \in E$.

Theorem # The set of ¹⁷⁴⁺¹ limit points of a bounded sequence is Compact set

Proof # Let E be the set of all limit points of a bounded sequence $\{a_n\}$.

Then E is closed and bounded. subset of \mathbb{R}

$\Rightarrow E$ is Compact

The Generalised Limits (Upper and lower limits)

We have discussed limit of a cgt sequence and have also proved that bounded monotone sequence always converge. There are sequences which are bounded but not monotone. Such sequences can converge but can equally well diverge as

$$\{(-1)^{n+1}\}$$

From this example we note that a general bounded sequence can diverge by oscillating between various limits. This oscillation suggests trigonometric function.

The \limsup & \liminf are defined for arbitrary (not necessarily cgt) sequences

If $\{a_n\}$ is bounded, then by B.W theorem $\{a_n\}$ has a cgt subsequence. The no $\limsup_{n \rightarrow \infty} a_n$ is the max value obtainable as the limit of cgt subsequence of $\{a_n\}$ i.e. maximum of all limit points/cluster points of a sequence and $\liminf_{n \rightarrow \infty} a_n$ is the minimum value obtainable as the limit of a cgt subsequence of $\{a_n\}$ i.e. min of all

Limit points/cluster points. 176

$\limsup_{n \rightarrow \infty} a_n$ & $\liminf_{n \rightarrow \infty} a_n$ are also denoted by $\overline{\lim}_{n \rightarrow \infty} a_n$ & $\underline{\lim}_{n \rightarrow \infty} a_n$

We discuss these limits under two categories

(a) For bounded sequences (b) for unbounded sequences.

Lim Sup & Lim Inf of Bounded Sequences

Let $\{a_n\}$ be a bounded sequence and E be the set of all limit points of $\{a_n\}$. Then

$$\limsup_{n \rightarrow \infty} a_n = \text{lub } E = \sup E$$

$$\liminf_{n \rightarrow \infty} a_n = \text{glb } E = \inf E$$

Thus $\limsup_{n \rightarrow \infty} a_n = \mu$

if for every $\epsilon > 0$

$$|a_n - \mu| < \epsilon$$

for infinitely many values of n and no number larger than μ has this property.

$$\liminf_{n \rightarrow \infty} a_n = \nu$$

if for every $\epsilon > 0$

$$|a_n - \nu| < \epsilon$$

for infinitely many values of n & no number less than ν has this property.

is true for every $\epsilon > 0$
 μ is a limit point of $\{a_n\}$
 $\mu \in E$
 $\mu \in E$.

Theorem # Let $\{a_n\}$ be a bounded sequence.
 Then $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$

Proof # Let E be the set of subsequential limits of $\{a_n\}$
 By definition

$$\inf E \leq \sup E$$

$$\Rightarrow \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

Theorem # A real no u is the limit superior of a bounded sequence $\{a_n\}$ iff

- (a) For each $\epsilon > 0$, $a_n > u - \epsilon$ for infinitely many values of n .
- (b) For each $\epsilon > 0$, $a_n < u + \epsilon$ for all except for finitely many values of n .

Proof # Necessity

Let u be \limsup of $\{a_n\}$ & let $\epsilon > 0$ be given.

$\therefore u$ is a limit point / cluster point of a_n

$\therefore a_n \in (u - \epsilon, u + \epsilon)$ for infinitely many values of n

In particular $a_n > u - \epsilon$ for infinitely many values of n

Again since u is the greatest limit point, $u + \epsilon$ is not a limit point and therefore

$a_n \geq u + \epsilon$ for only finitely many values of n

(if for $\delta m \epsilon > 0$, $a_n \geq u + \epsilon$ for infinite values of n , then $\{a_n\}$ will have limit point $p \geq u + \epsilon$)

$\therefore a_n < u + \epsilon$ for all except finite values of n

Remarks # (1) For a Cgt sequence all subsequences converge to same limit but a dgt bounded sequence has many Cgt subsequences. \limsup & \liminf give the behaviour of the set of limit points. This behaviour signifies how much the sequence $\{a_n\}$ can rise or fall when n is very large enough. The set of limit points for a Cgt sequence is not empty & is also bounded and hence lub, glb

(2) If $\{a_n\}$ is unbounded, $\limsup_{n \rightarrow \infty} a_n = \infty$
and $\liminf_{n \rightarrow \infty} a_n = -\infty$

Example (i) For sequence $\{(-1)^n\}$, the only limit points are $-1, 1$. So $E = \{-1, 1\}$

$$\liminf_{n \rightarrow \infty} a_n = -1 \quad \limsup_{n \rightarrow \infty} a_n = \sup E = 1$$

(ii) For sequence $\{\frac{1}{n}\}$, the only limit point is $\{0\}$. So $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = 0$

(iii) If $a_n = k \quad \forall n$

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = k$$

(iv) If $a_n = \begin{cases} 2 & \text{when } n \text{ is odd} \\ -n & \text{when } n \text{ is even} \end{cases}$

Then 2 is limit point of $\{a_n\}$ which is unbounded below. $\therefore \liminf_{n \rightarrow \infty} a_n = -\infty \quad \limsup_{n \rightarrow \infty} a_n = 2$

(v) For sequence $a_n = (-1)^n n \quad \forall n \in \mathbb{N}$

$$\liminf_{n \rightarrow \infty} a_n = -\infty \quad \limsup_{n \rightarrow \infty} a_n = \infty$$

term that follows

Sufficiency

178+12/29

Let u satisfies both conditions.
Given $\epsilon > 0$, $u - \epsilon < a_n$ for infinitely many values of n and $u + \epsilon > a_n$ for all except finitely many values of n

$\Rightarrow u - \epsilon < a_n < u + \epsilon$ for infinitely many values

$\Rightarrow u$ is a limit point of $\{a_n\}$

Now we show that no no greater than u can be limit point of $\{a_n\}$.

Let u' be any other limit point greater than u .
Let p, q be two numbers such that

$$u < p < u' < q$$

By 2nd condition, for each $\epsilon > 0$, $a_n < u + \epsilon$ for all except for finite values of n

Choosing $p - u = \epsilon > 0$, we have.

$a_n < p$ for all except finitely many values of n

and therefore (p, q) is a nbd of u' containing a_n for finitely many values of n . $\Rightarrow u'$ is not a limit point of $\{a_n\}$ and u is the greatest limit of $\{a_n\}$

Hence u is limit superior of $\{a_n\}$

Theorem # A real no l is the limit inferior of a bounded sequence $\{a_n\}$ iff the following are true

- (i) for each $\epsilon > 0$, $a_n \geq l - \epsilon$ for infinitely many values of n
- (ii) for each $\epsilon > 0$, $a_n < l + \epsilon$ for all except finitely many values of n

Proof Necessity

180

Let l be Limit inferior of $\{a_n\}$ and $\epsilon > 0$ be given

$\therefore l$ is Limit inferior of $\{a_n\}$

$\therefore l - \epsilon < a_n < l + \epsilon$ for infinitely many n

In particular

$a_n \geq l + \epsilon$ for infinitely many values

Again since l is the least limit point, $l - \epsilon$ is not a limit point and

$a_n \leq l - \epsilon$ for finitely many values

because if $\epsilon > 0$, $a_n \leq l - \epsilon$ for infinitely many values of n , then, then $\{a_n\}$ will have a limit point $p \leq l - \epsilon$

$\therefore a_n > l - \epsilon$ for all except finitely many values of n

Sufficiency #

Let us assume that l satisfies both the Conditions

Given $\epsilon > 0$, $a_n < l + \epsilon$ for infinite values of n and

$a_n > l - \epsilon$ for all except finite values of n

$\Rightarrow l - \epsilon < a_n < l + \epsilon$ for infinitely many values of n

$\therefore l$ is a limit point of $\{a_n\}$

We show that no number less than l can be a limit point of $\{a_n\}$

let l' be any number less than l

may

term

that

Let p, q be two ¹⁸¹ members such that
 $p < l' < q < l$

By 2nd condition, for each $\epsilon > 0$

$a_n > l - \epsilon$ for all except for finite values of n

\Rightarrow for $\epsilon = l - q > 0$, we have,

$a_n > l - \epsilon = l - (l - q) = q$ for all except finite values of n

$\Rightarrow (p, q)$ is a nbd of ~~any~~ l' containing a_n

for finitely many values of n

$\Rightarrow l'$ is not a limit point of $\{a_n\}$ so that
 l is the least limit point of $\{a_n\}$ and hence.

l is limit inferior of $\{a_n\}$

Theorem # A sequence $\{a_n\}$ converges to l iff

$$\lim_{n \rightarrow \infty} \sup a_n = \lim_{n \rightarrow \infty} \inf a_n = l$$

Proof # Let the sequence $\{a_n\}$ converges to l

Then given $\epsilon > 0$ \exists +ve integer m such that

$$|a_n - l| < \epsilon \quad \forall n \geq m$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m$$

Since the nbd $(l - \epsilon, l + \epsilon)$ of l contains a_n for infinitely many values of n and since ϵ is arbitrary therefore every nbd of l contains infinitely many terms of the sequence $\{a_n\}$

$\therefore l$ is a limit point of $\{a_n\}$

We show that l is only limit point of $\{a_n\}$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = l \Rightarrow \{a_n\} \text{ has only one cluster point}$$

Let l' be any number 182 other than l . Two cases arise

(i) $l < l'$

(ii) $l' < l$

Suppose $l < l'$. Let p, q, r be three numbers such that $p < l < q < l' < r$

$\therefore a_n \rightarrow l$

\therefore Every nbd of l contains a_n for all except finitely many values of n . In particular (p, q) for all except finitely many values of n

\Rightarrow The nbd (q, r) of l' contains a_n for almost finite values of n

$\Rightarrow l'$ cannot be a limit point of $\{a_n\}$

Similarly $l' < l$, l' is not a limit point of $\{a_n\}$

Thus l is only limit point of $\{a_n\}$

Hence $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$

Converse Let $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$

Let $\epsilon > 0$ be given.

$\therefore l = \limsup_{n \rightarrow \infty} a_n$

$\therefore a_n < l + \epsilon$ for all except finite values of n

$\Rightarrow \exists$ a pos integer m_1 s.t. $a_n < l + \epsilon$ if $n \geq m_1$

Again $\therefore l = \liminf_{n \rightarrow \infty} a_n$

$\therefore a_n > l - \epsilon$ for all except finitely many values of n

$\Rightarrow \exists$ a pos integer m_2 such that $a_n > l - \epsilon$ if $n \geq m_2$

- less than l

that follows

Let $m = \max(m_1, m_2)$, then
 $l - \epsilon < a_n < l + \epsilon \quad \forall n \geq m$
 $|a_n - l| < \epsilon \quad \forall n \geq m$
 $\Rightarrow a_n \rightarrow l$

2nd Definition of Limit Superior and Limit Inferior (for bounded sequence)

Let $\{a_n\}$ be bounded sequence and bounded above by K . Then for each $n \in \mathbb{N}$ the set $S_n = \{a_n, a_{n+1}, \dots\}$ is bounded above by K . By L.U.B. axiom S_n has lub M_n .
 i.e.

$$M_n = \sup S_n = \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}$$

clearly $M_n \geq M_{n+1} \quad \forall n \in \mathbb{N}$

Thus the sequence $\{M_n\}$ being decreasing sequence either converges or diverges to $-\infty$.

If $\{M_n\}$ is convergent, then limit M_n is called limit superior of $\{a_n\}$ i.e.

$$\lim_{n \rightarrow \infty} \sup a_n = \lim_{n \rightarrow \infty} M_n$$

if $\{a_n\}$ not bounded above we can let $M_n = \infty$

~~If $\{M_n\}$ is divergent, then $\lim_{n \rightarrow \infty} \sup a_n = +\infty$~~

Let $\{a_n\}$ be bounded below by k . Then each n
 $S_n = \{a_n, a_{n+1}, \dots\}$ is bounded below by k for each n

$\Rightarrow S_n$ has g.l.b $\frac{L+B}{2}$
 clearly $m_n \leq m_{n+1} \quad \forall n \in \mathbb{N}$.

Thus sequence $\{m_n\}$ being an increasing sequence is either convergent or diverges to $+\infty$.
 If $\{m_n\}$ is cgt, then

$$\lim_{n \rightarrow \infty} \inf a_n = \lim_{n \rightarrow \infty} m_n = \lim_{n \rightarrow \infty} \inf S_n$$

If $\{a_n\}$ is not bounded below we define $\liminf a_n = -\infty$
Theorem # (a) If a sequence $\{a_n\}$ is such that

$$\lim_{n \rightarrow \infty} a_n = \lim a_n = \infty$$

, then $\{a_n\}$ diverges to ∞

(b) If a sequence $\{a_n\}$ is such that

$$\lim a_n = \lim a_n = -\infty, \text{ then } \{a_n\}$$

diverges.

Proof # (a) Let $M_n = \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}$
 and $m_n = \inf \{a_n, a_{n+1}, a_{n+2}, \dots\}$ $\forall n$.

$$\text{Then } m_n \leq a_n \leq M_n \quad \forall n. \quad \text{--- (1)}$$

$$\text{Since } \lim_{n \rightarrow \infty} m_n = \lim_{n \rightarrow \infty} a_n = \infty \quad \& \quad \lim_{n \rightarrow \infty} M_n = \lim_{n \rightarrow \infty} a_n = \infty$$

$$\text{from (1)} \quad \lim_{n \rightarrow \infty} a_n = \infty$$

The sequence diverges to ∞

$$(b) \quad \therefore \lim_{n \rightarrow \infty} m_n = \lim_{n \rightarrow \infty} a_n = -\infty = \lim a_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = -\infty$$

$$\Rightarrow \{a_n\} \text{ diverges to } -\infty$$

$$\Rightarrow \exists n \quad a_n > l - \epsilon$$

term

term

that

Examples ¹⁸⁵

Q.1 Give examples of sequences having
 (i) no cluster point (ii) one cluster point
 (iii) Two cluster points (iv) infinitely many cluster points.

Sol (i) sequence $\{n\}$ has no cluster point
 (ii) The sequence $\{1/n\}$ has one cluster point
 (iii) The sequence $\{(-1)^n\}$ has two cluster points namely $-1, 1$
 (iv) The sequence $\{2, 1+\frac{1}{2}, 2+\frac{1}{2}, 1+\frac{1}{3}, 2+\frac{1}{3}, 3+\frac{1}{3}, 1+\frac{1}{4}, 2+\frac{1}{4}, 3+\frac{1}{4}, \dots\}$ has infinitely many cluster points. Every natural no is a cluster point

Q.2 Find cluster points of the sequences defined by n th term

- (i) $(-1)^n$ (ii) 5 (iii) $\frac{(-1)^n}{n}$
 (iv) n (v) $(-1)^n(1+\frac{1}{n})$ (vi) $(1+\frac{1}{n})^{n+1}$
 (vii) $1+\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$

Sol (i) $a_n = (-1)^n = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}$

$\{a_n\}$ has two cluster points.

(ii) $a_n = 5$ is constant sequence converging to 5
 $\Rightarrow \{a_n\}$ has only one cluster point

(iii) $a_n = \frac{(-1)^n}{n} = \begin{cases} -\frac{1}{n} & \text{when } n \text{ is odd} \\ \frac{1}{n} & \text{when } n \text{ is even} \end{cases}$

$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \{a_n\}$ has only one cluster point

(iv) for any $l \in \mathbb{R}$ ^{1.86} $(l - \frac{1}{4}, l + \frac{1}{4})$ contains at most one term of the sequence $\{n\}$
 $\therefore l \in \mathbb{R}$ is not a limit point of $\{a_n\}$
 \Rightarrow The sequence $\{n\}$ has no limit point

(v)

$$a_n = (-1)^n \left(1 + \frac{1}{n}\right) = \begin{cases} -(1 + \frac{1}{n}) & n \text{ odd} \\ (1 + \frac{1}{n}) & n \text{ even} \end{cases}$$

\Rightarrow Sequence $\{a_n\}$ has two cluster points $-1, 1$

(vi)

Here

$$a_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) = e \cdot 1 = e$$

\Rightarrow Sequence $\{a_n\}$ converges to e and has one cluster point

(vii)

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

The sequence converges to e

\therefore The sequence has only one cluster point

Q# Find the Limit Superior and Limit inferior of each of the following sequences.

(i) $\{1, 3, 5, 1, 3, 5, 1, 3, 5, \dots\}$

(ii) $\{1, 5, 17, 19, 1, 5, 17, 19, 1, 5, 17, 19, \dots\}$

(iii) $\{a_n\}$ where $a_n = \sin \frac{n\pi}{3}$

(iv) $\{a_n\}$ where $a_n = (-2)^n \left(1 + \frac{1}{n}\right)$

(v) $\{a_n\}$ where $a_n = (-10)^n \left(1 + \frac{1}{n}\right)^2$

(vi) $\{a_n\}$ where $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$

$\Rightarrow \lim_{n \rightarrow \infty} a_n = -$

$\Rightarrow \{a_n\}$ diverges to $-\infty$

$\Rightarrow \exists \dots a_n \rightarrow l - \epsilon$

(vii) $\{a_n\}$ $a_n = \frac{187}{(1+\frac{1}{n})^{n+1}}$

(viii) $\{a_n\}$ $a_n = (-1)^n (2^n + 3^n)$

$1 \quad 4 \quad 7$
 $a_m = 1 + (m-1) \cdot 3$
 $= 3m - 2$

Sol (i) $a_n = \begin{cases} 1 & \text{if } n = 3m-2 \\ 3 & \text{if } n = 3m-1 \\ 5 & \text{if } n = 3m \end{cases} \quad m \in \mathbb{N}$

The set of cluster points of $\{a_n\} = E = \{1, 3, 5\}$

$\lim_{n \rightarrow \infty} a_n = \text{Max}\{1, 3, 5\} = 5$

$\lim_{n \rightarrow \infty} a_n = \text{min}\{1, 3, 5\} = 1$

(ii) The set of cluster points $= E = \{1, 5, 17, 19\}$

$\lim_{n \rightarrow \infty} a_n = \text{Max } E = 19$ $\lim_{n \rightarrow \infty} a_n = \text{Min } E = 1$

(iii) $a_n = \sin \frac{n\pi}{3} = \begin{cases} 0 & \text{if } n = 3m \\ \frac{\sqrt{3}}{2} & \text{if } n = 6m-5 \text{ or } 6m-4 \\ -\frac{\sqrt{3}}{2} & \text{if } n = 6m-2 \text{ or } 6m-1 \end{cases} \quad \text{where } m \in \mathbb{N}$

The set of cluster points of $\{a_n\} = E = \{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\}$

$\Rightarrow \lim_{n \rightarrow \infty} a_n = \frac{\sqrt{3}}{2}$ $\lim_{n \rightarrow \infty} a_n = -\frac{\sqrt{3}}{2}$

(iv) Here $\{a_n\}$ converges to 0
The set of cluster points $= \{0\}$
 $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} a_n$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} \frac{188}{a_n}$$

(V) Here $a_n = (-1)^n \left(1 + \frac{1}{n}\right)^2 = \begin{cases} -\left(1 + \frac{1}{n}\right)^2 & n \text{ odd} \\ \left(1 + \frac{1}{n}\right)^2 & n \text{ even} \end{cases}$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = \infty$$

$$E = \{-\infty, +\infty\}$$

$$\lim_{n \rightarrow \infty} a_n = \infty$$

$$\lim_{n \rightarrow \infty} a_n = -\infty$$

(VI) $a_n = (-1)^n \left(1 - \frac{1}{n}\right) = \begin{cases} 1 - \frac{1}{n} & n \text{ even} \\ -1 + \frac{1}{n} & n \text{ odd} \end{cases}$

$$E = \{1, -1\}$$

$$\lim_{n \rightarrow \infty} a_n = -1$$

$$\lim_{n \rightarrow \infty} a_n = 1$$

(VII) $a_n = \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)$

$$\lim_{n \rightarrow \infty} a_n = e \times 1 = e$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = e$$

(VIII) $a_n = (-1)^n (2^n + 3^n) = \begin{cases} -(2^n + 3^n) & n \text{ is odd} \\ (2^n + 3^n) & n \text{ even} \end{cases}$

$$E = \{-\infty, \infty\} \Rightarrow \lim_{n \rightarrow \infty} a_n = \infty \quad \lim_{n \rightarrow \infty} a_n = -\infty$$

(VI) $\{a_n\}$ where $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$

$$\Rightarrow \{a_n\} \text{ diverges to } -\infty$$

$$\Rightarrow \exists \epsilon > 0 \text{ such that } a_n > 1 - \epsilon$$

Theorem # For ¹⁸⁹ a sequence $\{a_n\}$ prove that

$$\lim_{n \rightarrow \infty} a_n \leq \overline{\lim}_{n \rightarrow \infty} a_n.$$

Proof # If $\{a_n\}$ is unbounded, then
 either $\lim_{n \rightarrow \infty} a_n = \infty$ or $\lim_{n \rightarrow \infty} a_n = -\infty$

and hence there is nothing to prove.

Let a_n be a bounded sequence.

Let $m_n = \text{glb}\{a_n, a_{n+1}, a_{n+2}, \dots\}$

$M_n = \text{lub}\{a_n, a_{n+1}, a_{n+2}, \dots\} \quad \forall n \in \mathbb{N}.$

Then $m_n \leq M_n.$

$\Rightarrow \lim_{n \rightarrow \infty} m_n \leq \lim_{n \rightarrow \infty} M_n$

$\Rightarrow \lim_{n \rightarrow \infty} a_n \leq \overline{\lim}_{n \rightarrow \infty} a_n$

Theorem # If $\{a_n\}$ & $\{b_n\}$ are two sequences
 such that $a_n \leq b_n \quad \forall n \in \mathbb{N}$, then

(i) $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n.$ (ii) $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$

Proof # Let $M_n = \text{lub}\{a_n, a_{n+1}, a_{n+2}, \dots\}$

$M'_n = \text{lub}\{b_n, b_{n+1}, \dots\}$

$m_n = \text{glb}\{a_n, a_{n+1}, \dots\}$

$m'_n = \text{glb}\{b_n, b_{n+1}, \dots\}$

$\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$

$$\begin{aligned}
 & \therefore a_n \leq b_n \quad \forall n \\
 & \therefore M_n \leq M'_n \quad \& \quad m_n \leq m'_n \quad \forall n \\
 & \Rightarrow \lim_{n \rightarrow \infty} M_n \leq \lim_{n \rightarrow \infty} M'_n \quad \& \quad \lim_{n \rightarrow \infty} m_n \leq \lim_{n \rightarrow \infty} m'_n \\
 & \Rightarrow \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n \quad \& \quad \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n
 \end{aligned}$$

Theorem # If $\{a_n\}$ & $\{b_n\}$ are bounded sequences, then show that

$$\begin{aligned}
 (i) \quad & \lim_{n \rightarrow \infty} (a_n + b_n) \leq \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \\
 (ii) \quad & \lim_{n \rightarrow \infty} (a_n + b_n) \geq \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n
 \end{aligned}$$

Proof # Let $M_n = \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}$
 $M'_n = \sup \{b_n, b_{n+1}, \dots\}$
 $m_n = \inf \{a_n, a_{n+1}, \dots\}$
 $m'_n = \inf \{b_n, b_{n+1}, \dots\}$

$\therefore \{a_n\}$ & $\{b_n\}$ are bounded.

$\therefore \{a_n + b_n\}$ is bounded.

$$\begin{aligned}
 \sup \{a_n + b_n, a_{n+1} + b_{n+1}, \dots\} & \leq \sup \{a_n, a_{n+1}, \dots\} \\
 & \quad + \sup \{b_n, b_{n+1}, \dots\} \\
 & = M_n + M'_n
 \end{aligned}$$

$$\inf \{a_n + b_n, a_{n+1} + b_{n+1}, \dots\} \geq \inf \{a_n, a_{n+1}, \dots\} + \inf \{b_n, b_{n+1}, \dots\}$$

$$\begin{aligned}
 (i) \quad \lim_{n \rightarrow \infty} (a_n + b_n) & = \lim_{n \rightarrow \infty} \sup \{a_n + b_n, a_{n+1} + b_{n+1}, \dots\} \\
 & \leq \lim_{n \rightarrow \infty} (M_n + M'_n)
 \end{aligned}$$

$\{a_n\}$ diverges to $-\infty$ $\lim_{n \rightarrow \infty} a_n = -\infty$

$$a_n > l - \epsilon$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} M_n + \lim_{n \rightarrow \infty} M'_n \\
 &= \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \lim_{n \rightarrow \infty} (a_n + b_n) &\geq \lim_{n \rightarrow \infty} (m_n + m'_n) \\
 &\geq \lim_{n \rightarrow \infty} m_n + \lim_{n \rightarrow \infty} m'_n \\
 &= \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n
 \end{aligned}$$

Note(1) By Combining the above two

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n &\leq \lim_{n \rightarrow \infty} (a_n + b_n) \leq \overline{\lim}_{n \rightarrow \infty} (a_n + b_n) \\
 &\leq \overline{\lim}_{n \rightarrow \infty} a_n + \overline{\lim}_{n \rightarrow \infty} b_n
 \end{aligned}$$

(2) In certain cases strict inequalities may hold.

e.g

$$a_n = (-1)^n \quad \& \quad b_n = (-1)^{n+1}$$

$$\text{Then } \lim_{n \rightarrow \infty} a_n \neq 1 \quad \lim_{n \rightarrow \infty} b_n \neq 1$$

$$\therefore a_n + b_n = 0 \quad \forall n$$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_n + b_n) = 0 < \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\& \quad \lim_{n \rightarrow \infty} (a_n + b_n) > \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

Theorem # If $\{a_n\}$ is ¹⁹² a bounded sequence, then

- (i) $\lim_{n \rightarrow \infty} (-a_n) = - \lim_{n \rightarrow \infty} a_n$
- (ii) $\lim_{n \rightarrow \infty} (-a_n) = - \lim_{n \rightarrow \infty} a_n$
- (iii) $\lim_{n \rightarrow \infty} (\lambda a_n) = \lambda \lim_{n \rightarrow \infty} a_n \quad \lambda > 0$
- (iv) $\lim_{n \rightarrow \infty} (\lambda a_n) = \lambda \lim_{n \rightarrow \infty} a_n \quad \lambda > 0$
- (v) $\lim_{n \rightarrow \infty} (\lambda a_n) = \lambda \lim_{n \rightarrow \infty} a_n \quad \lambda < 0$

Proof # $\because \{a_n\}$ is bounded.
 $\therefore \{-a_n\}$ & $\{\lambda a_n\}$ are also bounded

$$\begin{aligned} \text{(i)} \quad \lim_{n \rightarrow \infty} (-a_n) &= \lim_{n \rightarrow \infty} \sup \{-a_n, -a_{n+1}, \dots\} \\ &= \lim_{n \rightarrow \infty} -\inf \{a_n, a_{n+1}, \dots\} \\ &= - \lim_{n \rightarrow \infty} a_n \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \lim_{n \rightarrow \infty} (-a_n) &= \lim_{n \rightarrow \infty} \inf \{-a_n, -a_{n+1}, \dots\} \\ &= \lim_{n \rightarrow \infty} -\sup \{a_n, a_{n+1}, \dots\} \\ &= - \lim_{n \rightarrow \infty} \sup \{a_n, a_{n+1}, \dots\} \\ &= - \lim_{n \rightarrow \infty} a_n \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \lim_{n \rightarrow \infty} (\lambda a_n) &= \lim_{n \rightarrow \infty} \sup \{\lambda a_n, \lambda a_{n+1}, \dots\} \\ &= \lambda \lim_{n \rightarrow \infty} \sup \{a_n, a_{n+1}, \dots\} \\ &= \lambda \lim_{n \rightarrow \infty} a_n \end{aligned}$$

Try others.

$$\leq \lim_{n \rightarrow \infty} (a_n)$$

$\{a_n\}$ converges to $-\infty$

$$a_n > l - \epsilon$$

that follows

193
Subsequential Limit

A real no l is called a subsequential limit of a sequence $\{a_n\}$ if there is a subsequence of $\{a_n\}$ converging to l .

A subsequential limit of a sequence $\{a_n\}$ is also called a cluster point or limit point of the sequence.

Theorem # A real no l is called a subsequential limit of the sequence $\{a_n\}$ iff each neighbourhood $(l-\epsilon, l+\epsilon)$, $\epsilon > 0$ of l contains infinitely many terms of $\{a_n\}$ (ie iff l is a cluster point of a_n)

Proof # Let l be a subsequential limit of $\{a_n\}$. Then \exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converging to l .

Given $\epsilon > 0$ \exists even integer k_0 such that
 $|a_{n_k} - l| < \epsilon \quad \forall k \geq k_0$

$\Rightarrow a_{n_k} \in (l-\epsilon, l+\epsilon) \quad \forall k \geq k_0$

\Rightarrow Infinitely many terms of sequence $\{a_{n_k}\}$ & hence of $\{a_n\}$ lie in $(l-\epsilon, l+\epsilon)$

Converse Let each nbd of $(l-\epsilon, l+\epsilon)$ of l contains infinitely many terms of $\{a_n\}$

Then $a_n \in (l-\epsilon, l+\epsilon)$ for infinitely many values of n
In particular for $\epsilon = \frac{1}{n}$

$a_n \in (l-\frac{1}{n}, l+\frac{1}{n}) = I_n \quad " \quad " \quad "$

choose $a_{n_1} \in (l-1, l+1)$. ¹⁹⁴ Then $\exists n_2 > n_1$ such that
 $a_{n_2} \in (l-1/2, l+1/2) = I_2$

Continuing like this \exists a natural no n_{k_0} s. that
 $n_{k_0} > \dots n_2 > n_1$ & $a_{n_{k_0}} \in (l - \frac{1}{k_0}, l + \frac{1}{k_0})$
 $= I_{k_0}$

Again continuing in this way get
a subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Now for all $n_k \geq n_{k_0}$, we have $k \geq k_0$

$$\Rightarrow \frac{1}{k} \leq \frac{1}{k_0} \quad \& \quad -\frac{1}{k} \geq -\frac{1}{k_0}$$

$$\Rightarrow l + \frac{1}{k} \leq l + \frac{1}{k_0} \quad \& \quad l - \frac{1}{k} \geq l - \frac{1}{k_0}$$

$$\Rightarrow (l - \frac{1}{k}, l + \frac{1}{k}) \subset (l - \frac{1}{k_0}, l + \frac{1}{k_0})$$

$$\Rightarrow I_k \subset I_{k_0} \quad \forall k \geq k_0 \quad \forall n_k \geq n_{k_0}$$

$$\Rightarrow \forall n_k \geq n_{k_0}, a_{n_k} \in I_k \Rightarrow a_{n_k} \in I_{k_0}$$

$$\Rightarrow a_{n_k} \in (l - \frac{1}{k_0}, l + \frac{1}{k_0}) \quad \forall n_k \geq n_{k_0}$$

$$\Rightarrow |a_{n_k} - l| < \frac{1}{k_0} = \epsilon \quad \forall n_k \geq n_{k_0}$$

$\Rightarrow \{a_{n_k}\}$ converges to l

$\Rightarrow l$ is a subsequential limit of the sequence.

By Muhammad Hussain Assistant
Professor Govt Asghar Mall
College Rawalpindi